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Final Report on Scoring Methods Study

Appendices

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ASD-TDR-63-17, Vol II
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FINAL REPORT ON SCORING METHODS STUDY
APPENDICES

March 1963

AFSC Project No. 7844

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ABSTRACT

This second volume of the final report on the Scoring Methods Study contains the appendices which support elements of the discussion presented in Volume I. These appendices, except Appendix 2, were originally separately issued as attachments to the monthly progress reports. They have been extensively revised and new material has been added. They are as follows:

1. Appendix 1 - Trajectory Analysis and Synthesis
2. Appendix 2 - Effects of Own-Ship Angular Motion on Relative Trajectories
3. Appendix 3 - Rate Characteristics of Linear Relative Trajectories
4. Appendix 4 - Minimum Data Requirements and Variance Considerations for One-, Two-, and Three-Station Measurements
5. Appendix 5 - Least-Squares Adjustment of Two-Station Angle-Only Position Fixes with Reliabilities of Adjusted Data
6. Appendix 6 - Application of Polynomial-Based Smoothing and Interpolating Formulas to Scoring Problems.

PUBLICATION REVIEW

This technical documentary report has been reviewed and is approved.

L. S. KARABLY
Lt Colonel, USAF
Chief, Target Development Laboratory

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APPENDIX 1

TRAJECTORY ANALYSIS AND SYNTHESIS

APPENDIX I

TRAJECTORY ANALYSIS AND SYNTHESIS

Introduction

The mission of the scoring methods study is to investigate those phenomena which may have potential in the development of new scoring techniques, and, ultimately, to develop concepts for new methods of determining (scoring) the relative trajectory of a munition with respect to its target. One of the problems associated with this study is that of determining how (or whether) useful trajectory information can be generated through full exploitation of existing mathematical techniques. More specifically, what are the characteristics of the relative trajectories and how can they be used advantageously in the development of a trajectory scorer? What are the limitations of the assumptions that it would be convenient to make?

This paper deals with the use of position, velocity, and acceleration data to give trajectory information or improvements over measured trajectories. The constant-relative-velocity assumption is investigated and the velocity and acceleration characteristics of a rocket are developed with atmospheric drag and gravity effects neglected. The problem of non-zero relative acceleration is considered; and a particularly useful method for smoothing, interpolating, extrapolating and (if desired) integrating or differentiating trajectory data is discussed briefly. The topics just mentioned are taken up in the order named after a preliminary discussion of trajectory synthesis.

Synthesis of Trajectories

The term "trajectory synthesis" will be used to mean the techniques and procedures by which observed data are used to reconstruct the path of an object over some prescribed interval of observation. This definition includes any operations needed to obtain values of the trajectory variables from the reduced data and any data-improvement activities such as smoothing, curve fitting, interpolation, and extrapolation.

For scoring applications, the desired result is the relative trajectory of the weapon with respect to its target. This can be obtained directly by observing the motion of the missile from a station or stations on the target, and vice versa, or by making some observations from each vehicle. The indirect procedure is to measure the trajectories of the target and the weapon with respect to an external reference such as a coordinate system attached to the earth. The data needed to establish such trajectories can be obtained from one or more stations external to the observed objects or by use of inertial

instruments (accelerometers or velocimeters) attached to the objects; in the latter case, additional observations will be needed to establish the initial conditions. The data taken may include signals which are directly related to angles, distances, or linear and angular velocities and accelerations as functions of time. Frequently the data are not obtained continuously but at discrete times which are usually and preferably equally spaced; the data taken may cover the entire trajectory of the weapon in the vicinity of the target or only portions of it.

A knowledge of the behavior of the trajectory variables including their ranges and rates of variation is essential to selecting a method of scoring and the instruments and procedures to be used. The coverage required, the accuracy and response time needed in the instruments, and even the data reduction and processing procedures are all functions of these trajectory characteristics. The complexity of the equipment, the amount of effort required in the processing, and of course the cost of the whole operation can be considerably reduced through knowledge of the simplifying assumptions which are acceptable in a given application.

The purpose of this paper is to demonstrate the probable characteristics of the several types of relative trajectories which are of interest in this study. The validity of the constant-relative-velocity assumption is considered for all cases, and the characteristics of such trajectories are discussed. Since the scoring requirements for future weapon-target intercepts do not define the tactics to be employed, a discussion of the velocity and acceleration characteristics of a rocket (with gravity and aerodynamic forces neglected) is provided in support of conclusions made earlier and for use in checking these conclusions when more definite information about the problems becomes available. Some remarks are made about trajectories with non-zero relative acceleration and about data processing operations (integration, interpolation, etc.) for relative trajectories. Errors in the data due to bending, own-ship angular motions, and the measuring instruments can obscure but obviously do not determine the true nature of the trajectory. The effects of these errors and the procedures needed to minimize these effects are the subjects of separate studies made under this contract. Although some remarks on differential errors are made in this paper, it will usually be assumed that the data are measured without error with respect to a non-rotating (sometimes referred to as inertial) coordinate system.

A trajectory is established when the position of a moving point is known as a function of time. In general, three position coordinates are determined to a certain accuracy (directly or indirectly) at discrete points in time. These time-position values obtained from the original data may be smoothed, extrapolated, and interpolated. The desired functional relation may be obtained by means of graphs or by curve fitting.

The desired trajectory data may be obtained from successive observations of position as a function of time or, if suitable initial conditions can be obtained, synthesized from either velocity or acceleration data or both.

It is well known that the velocity may be obtained by the time integral of the acceleration:

$$\vec{V}(t) = \vec{V}(0) + \int_0^t \dot{\vec{V}}\, dt$$

Position may be obtained as the integral of the velocity or the second integral of the acceleration. For any position coordinate X,

$$\begin{aligned} X(t) &= X(0) + \int_0^t \dot{X}\, dt \\ &= X(0) + \int_0^t \left[\dot{X}(0) + \int_0^{t_2} \ddot{X}\, dt_1 \right] dt_2 \\ &= X(0) + t\dot{X}(0) + \int_0^t \left[\int_0^{t_2} \ddot{X}\, dt_1 \right] dt_2 \end{aligned}$$

If the velocities and accelerations are those of the weapon relative to the target, then the corresponding integrals for displacement represent displacements of the weapon relative to the target; similarly, the relative velocities are integrals of relative accelerations. If some point other than the target, such as the center of the earth or a point on its surface, is chosen for reference, the integrals again give relative velocities and displacements with respect to the reference coordinate system.

If position coordinates are to be obtained as functions of the time by integrating the velocity, it is necessary to know the initial values of the position coordinates and to obtain values for the velocity components as functions of time. In the simplest case, treated in the next section, the relative velocity is constant within the accuracy requirements, and the results of the integration can be given in closed form. For any position coordinate, X, the result would be

$$X(t) = X_o + \dot{X}_o t$$

where X_o is the initial condition and $\dot{X}_o$ is the constant velocity component. An important possible application for such linear relations is

the synthesis of relative trajectories for head-on attacks against an ICBM or satellite. If portions of the trajectory do not have a data record, linear extrapolation or interpolation can be used to compute these missing portions.

Similarly, at least two quantities are needed to obtain the velocity by integrating the acceleration. These are sufficient if the acceleration is constant, or practically constant. These quantities are the initial velocity and the average acceleration. Likewise, a knowledge of the initial position, initial velocity, and the average acceleration is sufficient to permit the computation of position as a function of the time by integrating the acceleration.

Just as the position could be determined by integrating the velocity, so may the velocity be determined by differentiating the position coordinates as functions of time. At least two positions as functions of the time are required for such a differentiation but more are preferable. A complete record of position vs time thus permits the computation of a complete record of both velocity and acceleration vs time. If initial conditions of position and velocity are known, a complete record of acceleration is sufficient to enable the computation of complete records of velocity and position as functions of time. In general terms, if sufficient initial conditions are known, then a record of position, velocity, or acceleration permits the computation of the remaining quantities in this group. This procedure may be required in computing the trajectory or portions of it in scoring applications.

A position coordinate obtained by integrating a velocity is subject to an error which is the integral of a velocity error. If the permissible error in such a position coordinate is ℓ and the maximum time period for the integration is Δt, then the maximum permissible constant error in velocity is $\ell/\Delta t$. Similarly, if position is obtained by integrating acceleration twice, then the maximum permissible constant error in acceleration is $2\ell/\Delta t^2$. Using a value of 100 ft for ℓ (as specified in in accuracy requirements for some tests), permits computation of numerical values for the maximum constant velocity and acceleration errors as functions of the integration time Δt. Such relations are shown by Figures 1 and 2.

Constant-Relative-Velocity Assumption

A relative trajectory has a greatly simplified analytic form when the assumption of linear relative trajectory (or constant relative velocity) is justified. This assumption will be justified if the displacement due to relative acceleration is sufficiently small to be ignored for the time interval of interest.

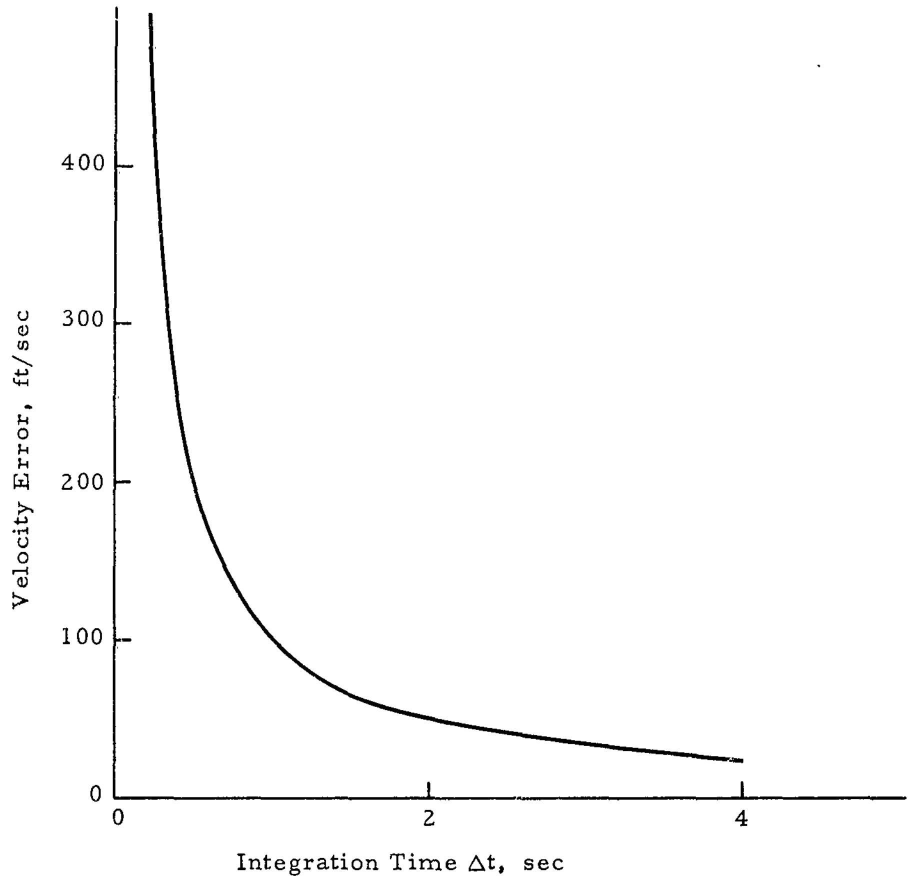

Figure 1. Constant Velocity Error for 100-ft Accumulated Position Error vs Integration Time Δt

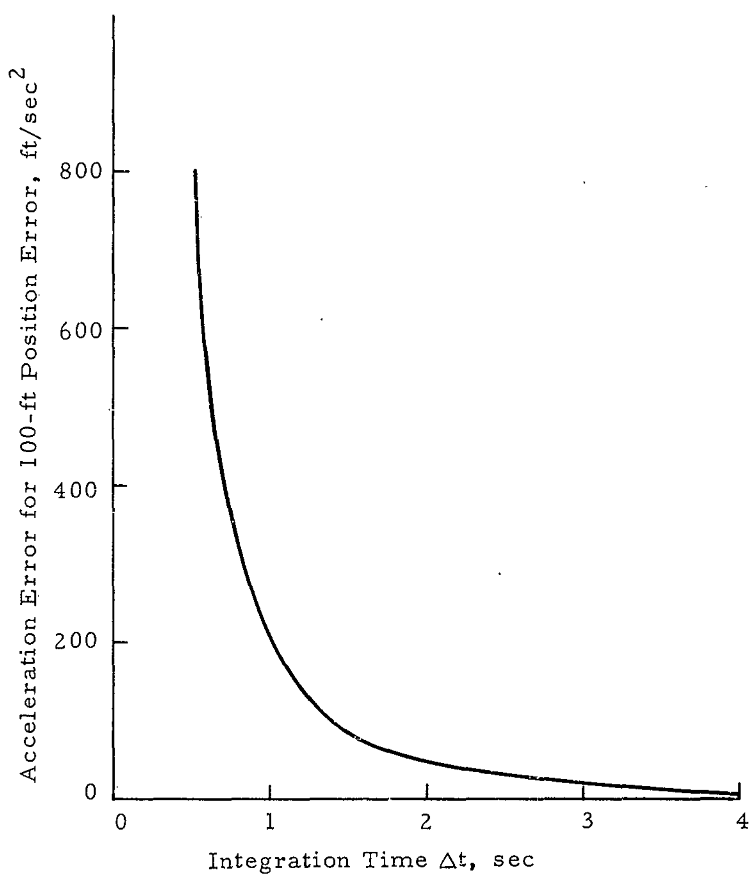

Figure 2. Constant Acceleration Error for 100-ft Accumulated Position Error vs Integration Time Δt

A typical scoring requirement specifies that an accuracy of $\pm \epsilon$ ft in position is desired for all ranges $R \leq R_{Max}$. Maximum and minimum closing rates are also given, and the time interval over which scoring is desired is then bounded by $2R_{Max}/\dot{R}_{Max} \leq \Delta t \leq 2R_{Max}/\dot{R}_{Min}$.

For the problem of scoring an attack on an air vehicle at any altitude between 70,000 and 200,000 ft, the requirements are, $\epsilon = 100$ ft, $R_{Max} = 3000$ ft, and $6000 \leq \dot{R} \leq 10,000$ ft/sec. The smallest time interval of interest for this type of attack is thus $6000/10,000 = 0.6$ sec; the largest time interval would be $6000/6000 = 1.0$ sec.

The requirements for scoring attacks on ICBM's are $R_{Max} = 2000$ ft and $\epsilon = 100$ ft. A study of the possible tactics for such attacks indicates that they are likely to be nose attacks, and the closing rates will thus be large. If the smallest closing rate is taken to be 10,000 ft/sec, the maximum time interval for scoring will be $4000/10,000 = 0.4$ sec.

The last problem to be considered is that of determining the trajectory of a missile fired from one satellite against a second satellite. It is specified that $R_{Max} = 500$ ft, and ϵ should be about 15 ft if the accuracy is to be comparable to that required for the air-vehicle problem. Closing rates would be enormous for head-on attacks in this case (of the order of 50,000 ft/sec), and it seems likely that the much lower rates associated with tail attacks would be desirable for accuracy. A value of 2000 ft/sec is not unreasonable for the minimum closing rate; this corresponds to a maximum time interval of $1000/2000 = 0.5$ sec.

The maximum error in position due to the assumption of constant velocity is the second integral of the relative acceleration over the time interval. If a maximum of Ng's relative acceleration is effective during the attack, the resulting range error will be no more than $\Delta R = 0.5\ Ng\ (\Delta t)^2$.

If the maximum position error is 100 ft,

$$100 \geq 16.1\ N(\Delta t)^2 \text{ or } N \leq 6.211/(\Delta t)^2$$

which relates the permissible average or constant relative acceleration to the time interval of the attack when the maximum error in position (due to the linear relative trajectory assumption) is taken to be 100 ft. A graph of this relation is shown in Figure 3. If the permitted error ϵ is not 100 ft, values from this curve should be multiplied by $(\epsilon/100)$.

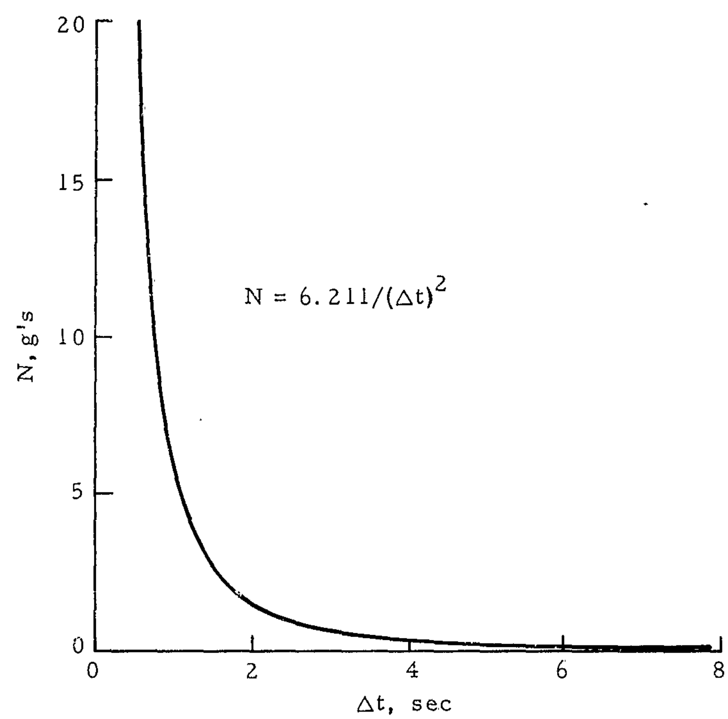

Figure 3. Relative Acceleration for 100-ft Accumulated Position Error vs Integration Time Δt

Since the time interval to be considered is related to R_{Max} and $\dot{R}$, N can be related to $\dot{R}$ instead of Δt. For 100-ft allowable error and for R_{Max} = 3000 ft, the relation is

$$100 \geq 16.1\,N(6000/\dot{R})^2 \text{ or } N \leq \dot{R}^2/5.796 \times 10^6$$

This relation is plotted in Figure 4. If ε is not 100 ft, values from this curve can be corrected as before by the factor (ε/100).

It will be noted that $\dot{R}$ is smallest and Δt is largest for tail attacks. These attacks take longer times to be completed, other factors being equal, so that relative acceleration acts over a longer time interval to cause displacement. Thus the permissible relative accelerations are smallest for tail attacks. This does not deny the validity of the assumption for certain tail chases, however. In particular, it is likely to be valid for the satellite-vs-satellite problem.

An average relative acceleration of 20 g's can cause a displacement of 100 ft in 0.5573 sec. If the range interval of interest is $R \leq 2000$ ft, then the linear-relative-trajectory assumption is valid for closing rates greater than or equal to 4000/0.5573 = 7177 ft/sec. The assumption of a linear relative trajectory over these ranges is thus apparently valid for all nose attacks on satellites and ICBM's.

If the target and weapon have constant velocities relative to a third observation station, they have constant velocities relative to each other. The converse is not necessarily true; they may have constant velocities relative to each other, to give a linear relative trajectory, while at the same time both target and weapon may have equal nonzero accelerations with respect to the third observation point. In this latter case, the trajectories of target and weapon with respect to this observation station are not linear.

Relative Motion of Munition With Respect to Target for Constant Relative Velocity

If the velocity of the weapon relative to the target can be well approximated as constant in magnitude and direction during an attack, the relative trajectory can be given as a linear function of time. If $\vec{R}$ is the range vector of the weapon with respect to the target, $\dot{\vec{R}}_o$ is its derivative in any inertial (non-rotating) coordinate system, and $\vec{R}_o$ is the initial value of $\vec{R}$, then

$$\vec{R}(t) = \vec{R}_o + \dot{\vec{R}}_o t \qquad (1)$$

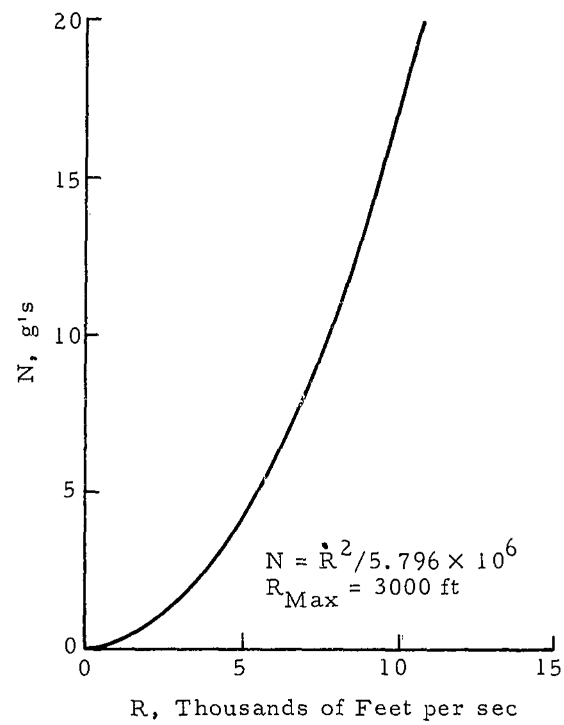

Figure 4. Relative Acceleration for 100-ft Accumulated Position Error vs Closing Rate, with Maximum Range 3000 ft

The relative coordinates in any inertial system are then given by

$$
\begin{aligned}
X(t) &= X_o + \dot{X}_o t \\
Y(t) &= Y_o + \dot{Y}_o t \\
Z(t) &= Z_o + \dot{Z}_o t
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
X_o &= X_{oW} - X_{oT} \\
\dot{X}_o &= \dot{X}_{oW} - \dot{X}_{oT}, \text{ etc.}
\end{aligned}
\tag{3}
$$

The subscripts W and T denote weapon and target values, respectively, and subscript o denotes the initial point. The length of the range vector is given by

$$R = (X^2 + Y^2 + Z^2)^{\frac{1}{2}}$$

$$R = \left[(X_o^2 + Y_o^2 + Z_o^2) + 2t(X_o\dot{X}_o + Y_o\dot{Y}_o + Z_o\dot{Z}_o) + t^2(\dot{X}_o^2 + \dot{Y}_o^2 + \dot{Z}_o^2)\right]^{\frac{1}{2}}$$

$$R = (\vec{R} \cdot \vec{R})^{\frac{1}{2}} = (R_o^2 + 2\vec{R}_o \cdot \dot{\vec{R}}_o t + \dot{R}_o^2 t^2)^{\frac{1}{2}} \tag{4}$$

If $\dot{\vec{R}}_o$ is constant and is aligned along $\vec{R}_o$, R is a linear function of the time, and a direct hit would result for positive closing rates; also the direction of $\vec{R}$ remains constant during the approach of the weapon to the target. On the other hand, if $\dot{\vec{R}}_o$ is not aligned with $\vec{R}_o$, the magnitude of R is not a linear function of the time, and

$$\frac{dR}{dt} = \frac{\vec{R}_o \cdot \dot{\vec{R}}_o + \dot{R}_o^2 t}{R} = \dot{\vec{R}}_o \cdot \vec{R}/R \tag{5}$$

The quantity dR/dt is of maximum magnitude at extreme range and decreases with time to the minimum range point, after which it again increases in magnitude. In all cases, dR/dt = 0 at the minimum range point. This gives a convenient method for finding the minimum range point. Since the numerator of equation (5) is zero at minimum range,

$$t_{R_{Min}} = -\vec{R}_o \cdot \dot{\vec{R}}_o / \dot{R}_o^2$$

$$\vec{R}_{Min} = \vec{R}_o - \dot{\vec{R}}_o (\vec{R}_o \cdot \dot{\vec{R}}_o) / \dot{R}_o^2 \qquad (6)$$

$$R_{Min} = \left[R_o^2 - (\vec{R}_o \cdot \dot{\vec{R}}_o)^2 / \dot{R}_o^2 \right]^{\frac{1}{2}}$$

Observe also that if $\dot{\vec{R}}_o$ is not aligned along $\vec{R}_o$, the relative range $\vec{R}$ is no longer constant in direction. The minimum angular rate of rotation of $\vec{R}$ then occurs at maximum range, and the maximum angular rate of rotation of $\vec{R}$ occurs at the minimum range. Since $\vec{R}$ is the sum of $\vec{R}_o$ and $\dot{\vec{R}}_o t$, the relative motion occurs in the plane of $\vec{R}_o$ and $\dot{\vec{R}}_o$. In this plane the relative velocity $\dot{\vec{R}}$ has components along $\vec{R}$ and normal to it, with the sums of the squares of these components being equal to the magnitude of $(\dot{\vec{R}} \cdot \dot{\vec{R}})$. At the point of closest approach $(\dot{\vec{R}} \cdot \vec{R}_{Min}) = 0$ so that the relative velocity is normal to $\vec{R}_{Min}$. The velocity normal to $\vec{R}$ may in general be represented as the vector cross product of the rotational rate vector of $\vec{R}$ and $\vec{R}$ itself; i.e., the component of $\dot{\vec{R}}$ normal to $\vec{R}$ is given by $\vec{\Omega}_R \times \vec{R}$, where $\vec{\Omega}_R$ is the vector angular rotation rate of $\vec{R}$ and is normal to both $\vec{R}$ and $\dot{\vec{R}}$. Actually,

$$\vec{\Omega}_R = \frac{\vec{R} \times \dot{\vec{R}}}{R^2} \qquad (7)$$

This angular rate has the maximum value

$$\left| \vec{\Omega}_R \right|_{Max} = \frac{\left| \dot{\vec{R}} \right|}{\left| \vec{R}_{Min} \right|} = \frac{\left| \dot{\vec{R}}_o \right|}{\left[R_o^2 - (\vec{R}_o \cdot \dot{\vec{R}}_o)^2 / \dot{R}_o^2 \right]^{\frac{1}{2}}} \qquad (8)$$

This maximum angular rate tends to ∞ as $|\vec{R}_{Min}|$ tends to zero, so that large angular rates are to be expected for near misses at the point of closest approach. The angular rate of $\vec{R}$, given by $\vec{\Omega}_R$, may be resolved into components in any convenient reference frame, and

azimuth and elevation rates can then be determined. Note that any component of $\vec{\Omega}_R$ must have absolute value $\leq \left|\vec{\Omega}_R\right|_{Max}$. These angular rates are those of the weapon as viewed from the target and expressed in an inertial reference frame. The angular rates of the weapon and of the target as seen from a third reference point may be considerably different from these, but they would be given by similar relations.

For such linear relative trajectories the direction of $\vec{R}$ rotates through almost 180° from the beginning of attack to final departure of the weapon from the immediate neighborhood of the target. If the weapon is to be observed from the target throughout the trajectory, an angular coverage of over 180° would be required; and if the side on which the weapon passes the target is not specified in advance, 360° coverage should be provided. However, if viewing the target up to the point of closest approach is sifficient, only half this angular coverage would be required. Extrapolation or integration could be used to compute portions of the trajectory outside the field of view.

The propagation of small errors in these vector and analytic relations may be studied by taking the differentials. Thus,

$$d\vec{R} = d\vec{R}_o + t d\dot{\vec{R}}_o + \dot{\vec{R}}_o \, dt \tag{9}$$

This equation gives the effects of small errors made in measuring $\vec{R}_o$, $\dot{\vec{R}}_o$, and t upon the resulting error in $\vec{R}$. Normally, dt will be negligibly small in comparison to other errors in scoring tests. Taking the components of $d\vec{R}$ gives

$$dX = d\left[(X_{oW} - X_{oT}) + t(\dot{X}_{oW} - \dot{X}_{oT})\right]$$

$$dX = dX_{oW} - dX_{oT} + (\dot{X}_{oW} - \dot{X}_{oT})\,dt + t(d\dot{X}_{oW} - d\dot{X}_{oT}) \tag{10}$$

with similar expressions for dY and dZ.

The error in the measured value of R is given by

$$dR = (Xdx + Ydy + ZdZ)/R \tag{11}$$

The error in $\vec{R}_{Min}$ computed from equation (6) is given approximately by

$$d\vec{R}_{Min} = d\vec{R}_o - d\left[\dot{\vec{R}}_o(\vec{R}_o \cdot \dot{\vec{R}}_o)/\dot{\vec{R}}_o \cdot \dot{\vec{R}}_o\right]$$

$$d\vec{R}_{Min} = d\vec{R}_o - \frac{d\dot{\vec{R}}_o(\vec{R}_o \cdot \dot{\vec{R}}_o) + \dot{\vec{R}}_o(d\vec{R}_o \cdot \dot{\vec{R}}_o) + \dot{\vec{R}}_o(\vec{R}_o \cdot d\dot{\vec{R}}_o)}{\dot{R}_o^2}$$

$$+ \frac{\dot{\vec{R}}_o(\vec{R}_o \cdot \dot{\vec{R}}_o)(2d\dot{\vec{R}}_o \cdot \dot{\vec{R}}_o)}{\dot{R}_o^4} \tag{12}$$

If the angular rotation rate $\vec{\Omega}_R$ is computed from equation (7), the differential error is given by

$$d\vec{\Omega}_R = \frac{d\vec{R} \times \dot{\vec{R}} + \vec{R} \times d\dot{\vec{R}}}{R^2} - \frac{2(\vec{R} \times \dot{\vec{R}})(\vec{R} \cdot d\vec{R})}{R^4} \tag{13}$$

Velocity and Acceleration of a Rocket (Atmospheric Drag and Gravity Effects Neglected)

Simple relations may be obtained to approximate the velocity and acceleration functions of a rocket when atmospheric drag and gravity effects are neglected. These relations will be valid when the thrust is large compared to the neglected effects. It will be assumed that the thrust is aligned with the velocity vector.

The general equation for the acceleration reduces in this case to

$$m\dot{V} = -Ng\,\dot{m} = \text{thrust} \tag{14}$$

where the dot denotes the time derivative, and

V = velocity, ft/sec

m = mass of rocket and remaining fuel, slugs

$\dot{m}$ = negative of fuel burning rate, slugs/sec

N = specific impulse, lb-sec/lb or sec

g = gravitational constant, ft/sec^2.

This equation may be integrated to give the velocity increment.

$$dV = -Ng \frac{dm}{m}$$

$$\Delta V = V - V_o = Ng \ln \frac{1}{1 - (m_f/m_o)} \tag{15}$$

where m_f is the mass of fuel expended, and m_o is the initial total mass of rocket and fuel.

Figure 5 is a plot of ΔV vs m_f/m_o for three values of N. Since equation (15) shows that the velocity increment is linear in N, the value of ΔV for any N is $(N/300)\Delta V(300, m_f/m_o)$ where $\Delta V(300, m_f/m_o)$ is the value of ΔV read from the curve for N = 300. Figure 6 is a plot of $\Delta V/Ng$ vs m_f/m_o, which is a dimensionless curve serving for all values of the variables.

If the burning rate $\dot{m}$ is constant, $m = m_o + \dot{m}t$, and equation (14) becomes

$$\dot{V} = \frac{-Ng\dot{m}}{m_o + \dot{m}t}$$

which can be written in the dimensionless form

$$-\frac{\dot{V}m_o}{Ng\dot{m}} = \frac{1}{1 + (\dot{m}/m_o)t} \tag{16}$$

Figure 7 shows a plot of $(-\dot{V}m_o/Ng\dot{m})$ vs $(-\dot{m}t/m_o)$. If N, m_o, and $\dot{m}$ are given, acceleration $\dot{V}$ is a function of motor burning time can be obtained from this curve for any time $0 \leq -\dot{m}t/m_o \leq m_f/m_o$ where m_f is the total mass of the fuel.

The velocity increment as a function of motor burning time for constant fuel burning rate may be similarly obtained. In this case equation (15) becomes

$$\Delta V \equiv V - V_o = Ng \ln \frac{1}{1 + \frac{\dot{m}t}{m_o}} \tag{17}$$

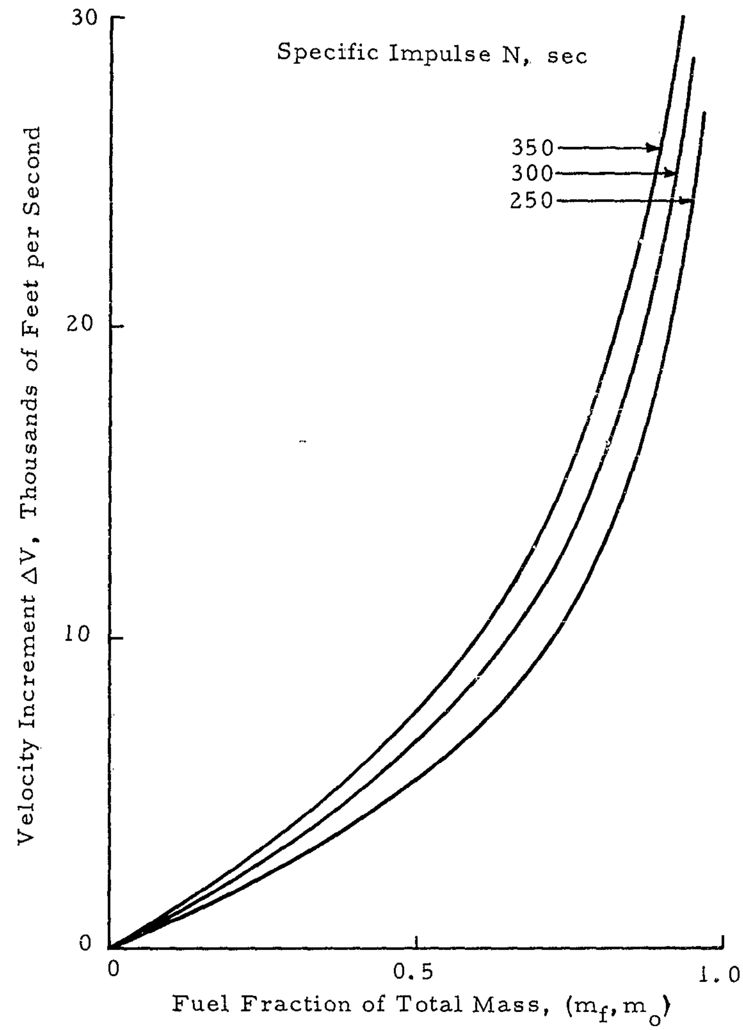

Figure 5. Rocket Velocity Increment as a Function of Specific Impulse and Fuel-Mass Ratio

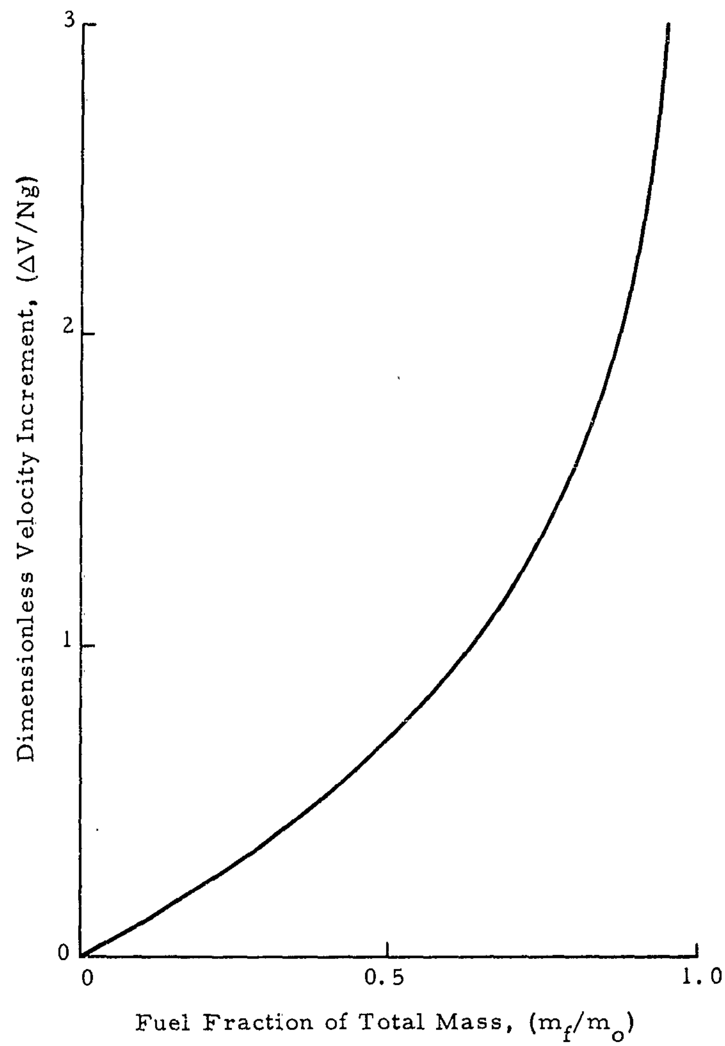

Figure 6. Rocket Dimensionless Velocity Increment as a Function of Fuel-Mass Ratio

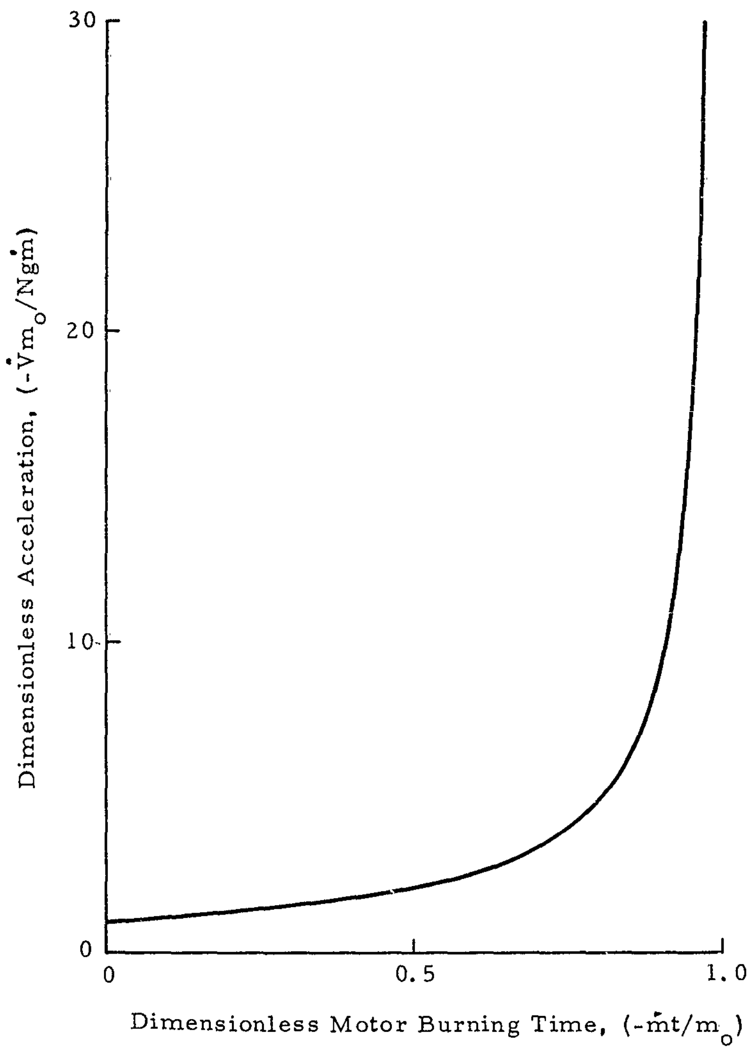

Figure 7. Dimensionless Acceleration as a Function of Motor Burning Time for Constant Burning Rate

Converting this equation to dimensionless form gives

$$\left(\frac{\Delta V}{Ng}\right) = \ln \frac{1}{1 + \left(\frac{\dot{m}t}{m_o}\right)} \tag{18}$$

Figure 8 is a plot of this relation, from which the velocity increment as a function of the burning time may be found. This curve is the same as Figure 6 with $-\dot{m}t$ substituted for m_f.

The displacement or change in position due to motor burning at a constant rate may be obtained by integrating equation (17) with respect to time. If S is the total displacement during motor burning, then the displacement due to acceleration is $S - V_o t$. Integration of equation (17) gives

$$\frac{-\dot{m}\,(S - V_o t)}{m_o Ng} = \left(1 + \frac{\dot{m}t}{m_o}\right) \ln\left(1 + \frac{\dot{m}t}{m_o}\right) - \frac{\dot{m}t}{m_o} \tag{19}$$

Figure 9 is a plot of this dimensionless displacement due to motor burning vs the dimensionless time $(-\dot{m}t/m_o)$.

Figure 10 is a dimensional plot of the displacement vs time for a hypothetical rocket with constant burning rate $\dot{m} = -m_o/150$ slugs/sec and specific impulse N = 300 sec. The curve ends at t = 135 sec corresponding to a fuel-mass ratio of 0.9. The acceleration increases with time to a maximum of 20 g's at burnout. It appears that a constant burning rate is not the most efficient way to program motor burning because larger displacements and the same final velocity result if the larger accelerations occur at the beginning of the flight.

Relative Trajectory with Non-Zero Relative Acceleration

The velocity and position of a weapon relative to a target may be obtained by integrating the relative acceleration. If the relative acceleration is n(t), then the relative velocity is given by

$$V(t) = V_o + \int_0^t n(t_1)\, dt_1 \tag{20}$$

The relative displacement of the weapon with respect to the target is then given by

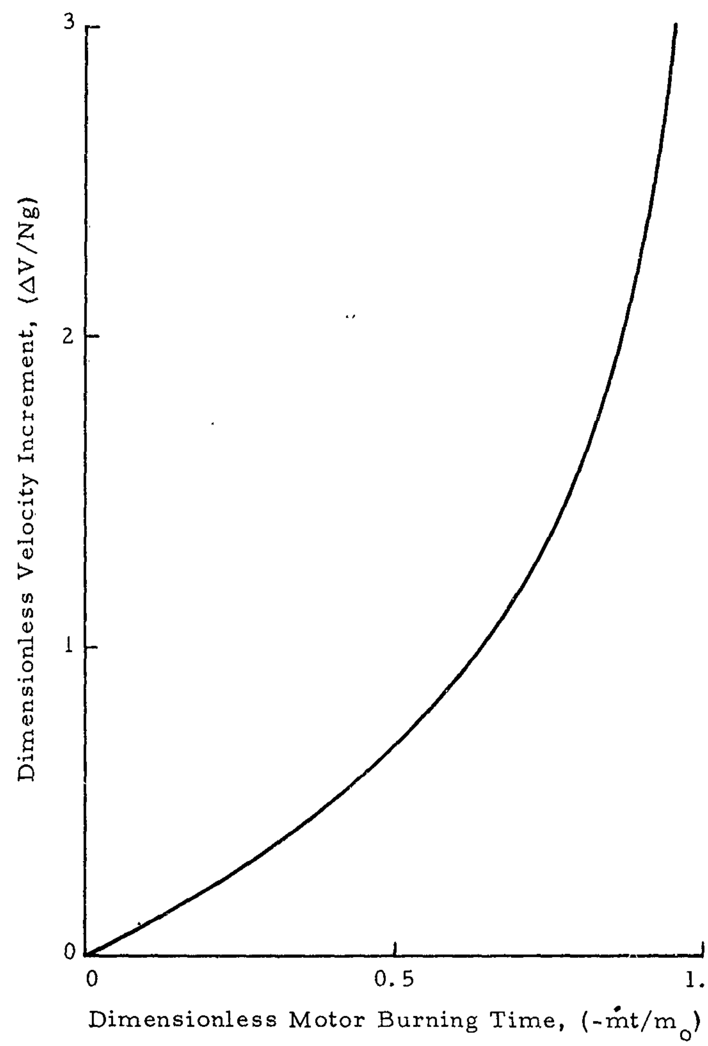

Figure 8. Dimensionless Velocity Increment as a Function of Dimensionless Motor Burning Time for Constant Burning Rate

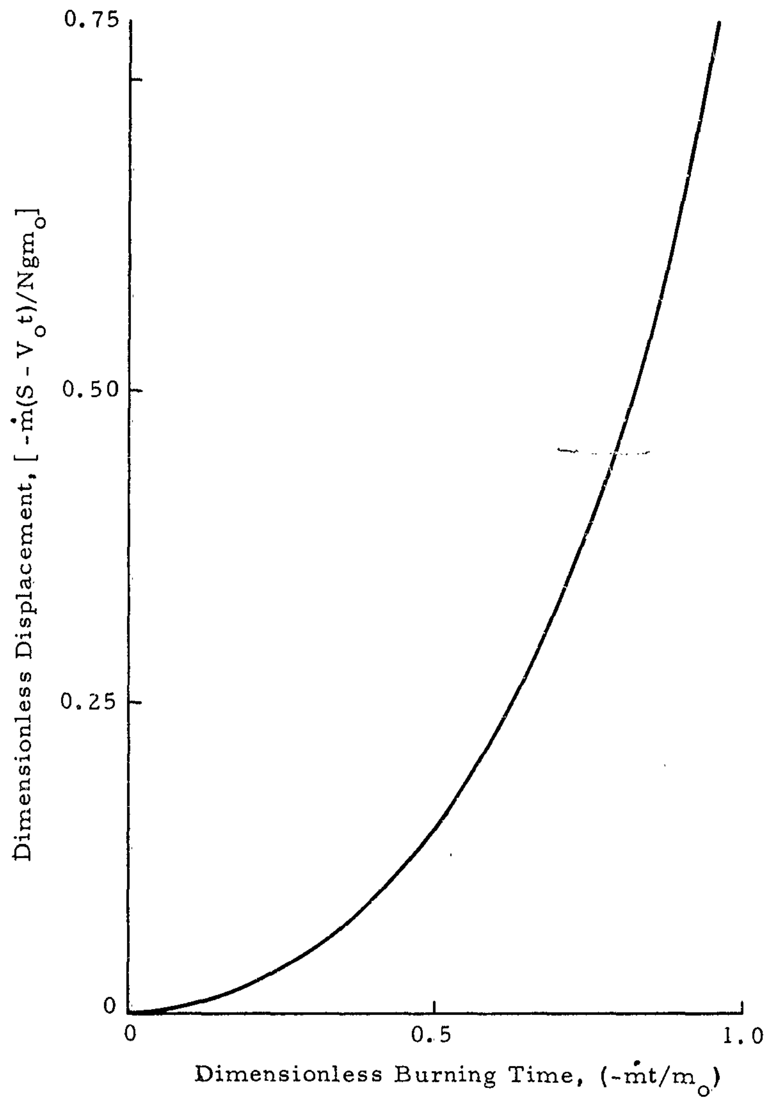

Figure 9. Dimensionless Displacement vs Dimensionless Motor Burning Time for Constant Burning Rate

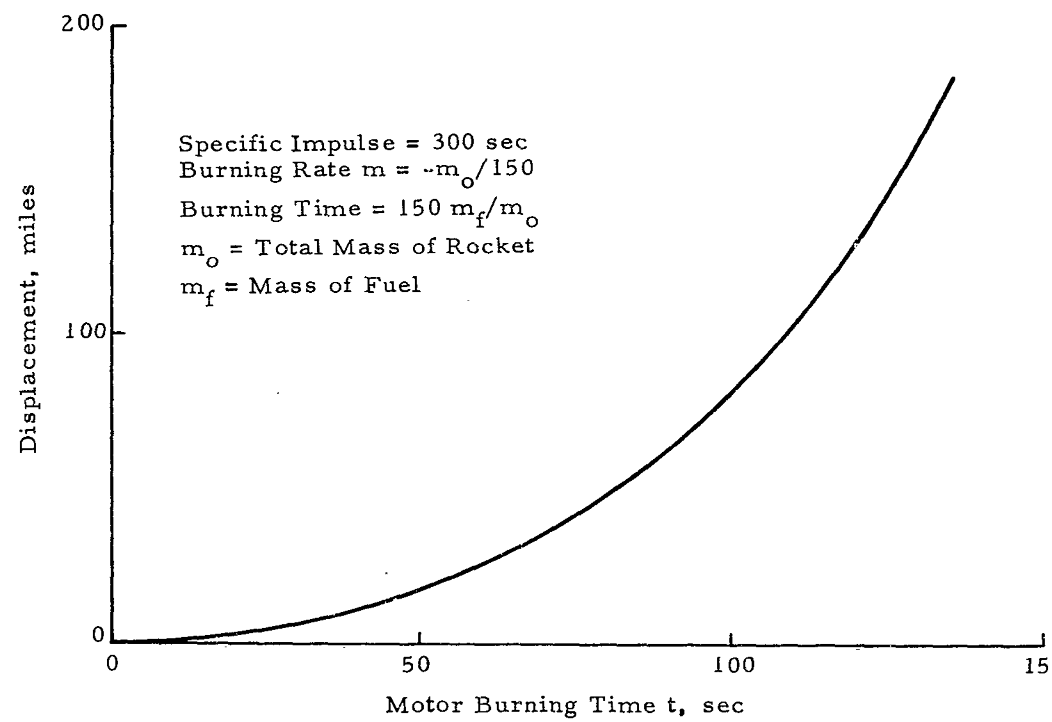

Figure 10. Example of Displacement vs Time for a Rocket With Constant Burning Rate

$$S(t) = S_o + \int_0^t V(t)\,dt = S_o + V_o t + \int_0^t dt_2 \int_0^{t_2} n(t_1)\,dt_1 \tag{21}$$

The relative velocity and displacement cannot generally be given as closed-form functions of time; but if the relative acceleration, n(t), is a linear function of time, the integrations can be performed. Let n(t) = A + Bt; then

$$V(t) = V_o + \int_0^t (A + Bt_1)\,dt_1 = V_o + At + Bt^2/2$$

$$S(t) = S_o + V_o t + At^2/2 + Bt^3/6 \tag{22}$$

Higher-order terms could be added to these equations, but it seems likely that the scoring encounters with which this study is concerned will be characterized by accelerations which are no worse than linear functions of time. If it can be assumed that the underlying function represented by the acceleration data is a polynomial in time of degree n or less, where n is known, the curve-fitting coefficients A, B, etc. do not have to be determined. Instead, smoothing and interpolation formulas can be developed which will perform the integration and give the trajectory directly. Reference 2 gives such formulas for polynomials of degree 3 or less.

Equations (22) represent the velocity and displacement for an acceleration which is a scalar function of the time. Similar relations would be obtained for each of the rectangular coordinate components if a vector acceleration were assumed.

Reference 1. "The Trajectory of a Rocket with Thrust," R. E. Struble, C. E. Stewart, and J. Granton, Jr., Jet Propulsion, Vol. 28 No. 7, July 1958, pp. 472-478.

Referense 2. Mathematical Processes: Sets of Coefficients for Data Processing Formulas, Evelyn Welborn, MPRL Report 475, Military Physics Research Laboratory, The University of Texas, Austin, Texas, 5 May 1959.

APPENDIX 2

EFFECTS OF OWN-SHIP ANGULAR MOTION ON RELATIVE TRAJECTORIES

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Introduction

The desired objective in taking data on missile-target scoring encounters is to secure enough information to establish within permissible error limits the trajectory of the weapon in the vicinity of and with respect to the target. It is highly desirable that this relative trajectory be referred to a stable coordinate system whose approximate absolute orientation is known. If the trajectory scoring method is based on assumptions about the nature of the relative trajectory, the data must be measured in (or at least be converted to) a stable system. A stable coordinate system is here defined as one which does not rotate appreciably during the scoring encounter.

It will be necessary in many instances to obtain trajectory data by use of instruments rigidly attached to the target or missile; the data taken will be referred to body axes of the parent object and will thus be taken with respect to a rotating coordinate system. The own-ship angular motions would then introduce errors into the data and into the relative trajectory obtained from the data; these errors must be taken into account and corrected for when they are sufficiently large.

The linear velocities and accelerations of the own-ship center of mass affect the relative trajectory, of course, but they do not of themselves introduce errors into observations taken from one of the objects. Errors will be introduced if instruments which are not intended to measure these quantities are sensitive to them and if the scoring method is based on invalid assumptions about the nature of the relative trajectory. These problems should be considered in the design of the scoring system but are not of present concern.

The purposes of this paper are to examine the effects of own-ship angular motions on the data, to establish criteria for neglecting these motions, and to discuss methods of taking them into account when they may not be neglected.

Trajectory Errors Due to Own-Ship Angular Motions

Figure 1 shows a stable rectangular Cartesian coordinate system X_o, Y_o, Z_o with origin O_o at the center of mass of the instrumented object; the rotating system X, Y, Z is fixed to the instrumented object and is shown with its origin O coinciding with O_o. The vector range from O to the observed object at point P is denoted by $\vec{R}$.

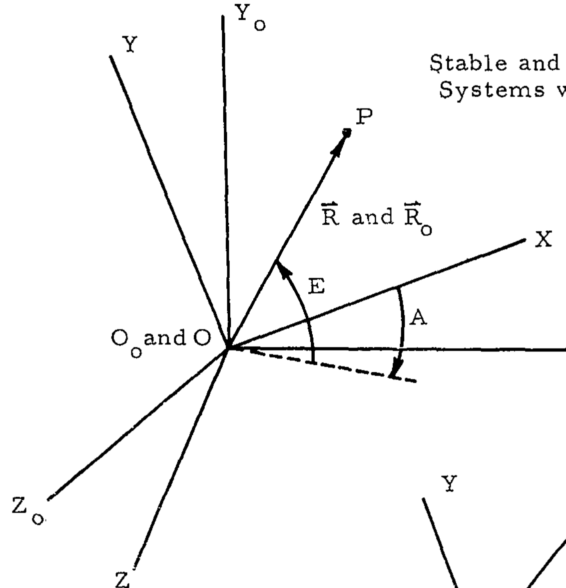

Figure 1

Stable and Rotating Coordinate Systems with Common Origin

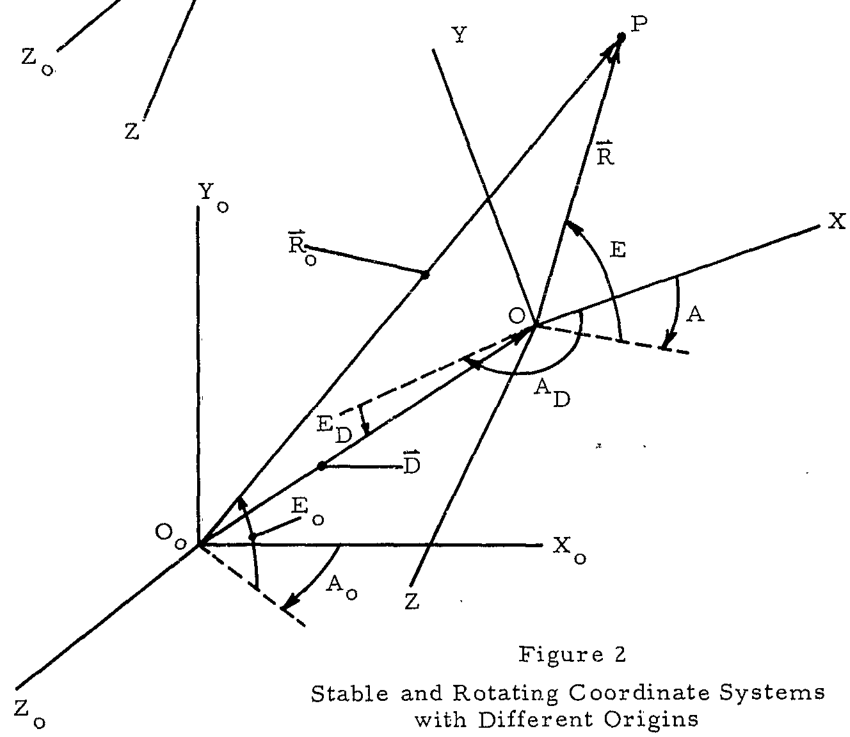

Figure 2

Stable and Rotating Coordinate Systems with Different Origins

The azimuth and elevation angles of $\vec{R}$, measured with respect to the rotating system, are denoted by A and E, respectively. Angle A is measured in the X-Z plane from the X axis to the plane of $\vec{R}$ and Y and is positive as shown; angle E is measured in the plane of $\vec{R}$ and Y from the X-Z plane to $\vec{R}$ and is positive as shown.

Figure 2 shows the situation when the origins are separated; here $\vec{D}$ is the vector from O_o to O, and $\vec{R}_o$ is the vector from O_o to P. Since measurements may be taken at two or more stations, it may be necessary in practice to consider two or more rotating systems: (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), etc. with origins at O_1, O_2, etc. If there is no bending, the rotating systems may all be parallel, but the vectors $\vec{D}_1$, $\vec{D}_2$, etc. will all be different. For simplicity, this discussion will be limited to a single rotating system; the extension to multiple systems is obvious.

Bending can occur as a result of aerodynamic loading, warping of the structure due to heating or cooling, etc.; it will have to be measured separately from the own-ship angular motions. The effects of bending will be to move the origin O and twist the X, Y, Z system. The coordinate transformation to correct for bending will therefore contain translation as well as rotation terms and could be developed from a diagram similar to that of Figure 2. The dynamic effects of bending may be small enough to neglect; but if the angular velocity and angular acceleration of the coordinate system are measured at or near origin O, no assumption need be made about this point. The change in the length of $\vec{D}$ is probably negligible, but it would have to be measured or determined from a calibration if it is not.

Values of the coordinates of point P in the X_o, Y_o, Z_o system will be determined by applying in succession the coordinate transformations for bending and for own-ship angular motions to the coordinates measured in the X, Y, Z system; but if the method of scoring assumes no bending, some preliminary calculations will be required. An example of this is the two-station angle-only triangulation system in which it is assumed that O is a fixed (known) distance from the second station and that X, Y, Z has a specified orientation with respect to the line connecting the stations; similar assumptions are made about the coordinate system at the second station. If significant bending occurs, it will be necessary to determine new standard orientations for the stations and then to correct the angular data and the length of the base line. After the corrected data have been used to compute the range, the coordinate transformations for bending and for own-ship angular motions would be applied, one after the other, to the values of X, Y, Z determined from the computed range R and the original uncorrected angles.

The angles used to specify the orientation of X, Y, Z with respect to X_o, Y_o, Z_o may vary from one application to the next, and it is therefore impossible to write specific coordinate transformations for general use. It can be said that if a stabilized platform is used, the orientation angles will be defined by the gimbal system; similarly, the bending angles are determined by the measurement system.

The coordinate transformation from the X, Y, Z system to the X_o, Y_o, Z_o system can be written in the form

$$
\begin{aligned}
X_o &= (C_{X_oX}X + C_{X_oY}Y + C_{X_oZ}Z) - C_{X_oX}D \cos E_D \cos A_D \\
Y_o &= (C_{Y_oX}X + C_{Y_oY}Y + C_{Y_oZ}Z) - C_{Y_oY} D \sin E_D \qquad (1) \\
Z_o &= (C_{Z_oX}X + C_{Z_oY}Y + C_{Z_oZ}Z) - C_{Z_oZ} D \cos E_D \sin A_D
\end{aligned}
$$

where C_{X_oX}, C_{X_oY}, etc. are direction cosines. Since the components of $\vec{D}$ in the X, Y, Z system are constant (neglecting bending which can be treated separately), it is more convenient to expand $\vec{D}$ in this system than in the X_o, Y_o, Z_o system. The general form of the coordinate transformation for bending will be identical to equations (1) except for changes in symbols. When these transformations are applied, the translation and rotation terms can be evaluated separately so that corrected components of $\vec{D}$ and $\vec{R}$ in the X_o, Y_o, Z_o system will be obtained.

The true velocity and acceleration vectors $\dot{\vec{R}} = d\vec{R}/dt$ and $\ddot{\vec{R}} = d^2\vec{R}/dt^2$ transform by equations (1) with the translation terms set equal to zero (by the product of two such transformations if bending occurs), but it should be noted that these vectors are not the ones which would be measured by instruments fixed in the X, Y, Z system. This can be shown by differentiating the equation

$$\vec{R}_o = \vec{D} + \vec{R}$$

It is convenient to consider the derivatives of $\vec{D}$ and $\vec{R}$ separately. Expressions for the derivatives of $\vec{R}$ will be developed first.

If $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors in the X, Y, Z system, the relative range vector $\vec{R}$ will be denoted by

$$\vec{R} = \vec{i}X + \vec{j}Y + \vec{k}Z \qquad (2)$$

Differentiation gives

$$\dot{\vec{R}} = (\vec{i}\dot{X} + \vec{j}\dot{Y} + \vec{k}\dot{Z}) + (\dot{\vec{i}}X + \dot{\vec{j}}Y + \dot{\vec{k}}Z) \tag{3}$$

$$\dot{\vec{R}} = \vec{V} + \vec{\omega} \times \vec{R} = \dot{\vec{R}}_o - \dot{\vec{D}} = \vec{V}_o - \dot{\vec{D}} \tag{4}$$

since it can be shown that $\dot{\vec{i}} = \vec{\omega} \times \vec{i}$, $\dot{\vec{j}} = \vec{\omega} \times \vec{j}$, and $\dot{\vec{k}} = \vec{\omega} \times \vec{k}$ where

$$\vec{\omega} = \vec{i}\,\omega_X + \vec{j}\,\omega_Y + \vec{k}\,\omega_Z \tag{5}$$

is the angular velocity of system X, Y, Z with respect to system X_o, Y_o, Z_o. Note that the equation

$$\dot{\vec{R}} \equiv \frac{d\vec{R}}{dt} = \frac{\partial \vec{R}}{\partial t} + \vec{\omega} \times \vec{R} = \left[\frac{\partial}{\partial t} + \vec{\omega} \times \right] \vec{R} \tag{6}$$

expresses a general operator relation applying to any vector measured with respect to the X, Y, Z system. Also note that the assumption that X_o, Y_o, Z_o is a stable system is unnecessary to this development; the equations apply to any two systems which have relative rotation.

The subscript "o" will be used to denote quantities measured with respect to the $X_o Y_o, Z_o$ system; thus $\vec{V}$ is the velocity of point P as observed in the X, Y, Z system, and $\vec{V}_o = \dot{\vec{R}}_o$ is the velocity of P as observed in the X_o, Y_o, Z_o system. For convenience, a dot over the symbol has been employed instead of the operator d/dt to denote the total time derivative (or the time rate of change of the quantity with respect to the desired reference system X_o, Y_o, Z_o); the operator $\partial/\partial t$ gives the time rate of change measured with respect to the X, Y, Z system.

Equation (4) shows that an observer in the X, Y, Z coordinate system who did not allow for own-ship rotation would report the velocity of point P as $\vec{V} = \vec{i}\dot{X} + \vec{j}\dot{Y} + \vec{k}\dot{Z}$, which is smaller than the desired vector $\vec{V}_o$ by the amount $\dot{\vec{D}} + \vec{\omega} \times \vec{R}$. If the same observer were to determine the range vector at any time t by integrating the velocity components $\dot{X}$, $\dot{Y}$, and $\dot{Z}$ with the required initial conditions, he would obtain $\vec{R} = \vec{i}X + \vec{j}Y + \vec{k}Z$, which is the correct value for his rotating coordinate system but is not equal to the desired vector $\vec{R}_o$.

The accumulated error in position would be given by the time

integral of $(\dot{\vec{D}} + \vec{\omega} \times \vec{R})$. If the integral of $\dot{\vec{D}}$ is taken to be negligible, the position error due to own-ship rotation would be bounded by $|\Delta\vec{R}| \leq |\omega_{Max} R_{Max} \Delta t|$, where Δt is the time interval involved. This bound for position error due to rotation of the reference coordinate system applies whether the relative trajectory is obtained by direct measurement or by integrating the relative velocity components. This bound would ordinarily be used to determine the error due to own-ship angular motion, but it could also be used to determine the error due to using a stabilization scheme which allows some rotation of the axes with respect to inertial space.

The effects of angular rotation of the reference coordinate system upon relative acceleration will now be examined. Differentiation of equation (3) gives

$$\ddot{\vec{R}} = (\vec{i}\ddot{X} + \vec{j}\ddot{Y} + \vec{k}\ddot{Z}) + 2(\dot{\vec{i}}\dot{X} + \dot{\vec{j}}\dot{Y} + \dot{\vec{k}}\dot{Z}) + (\ddot{\vec{i}}X + \ddot{\vec{j}}Y + \ddot{\vec{k}}Z)$$

or

$$\ddot{\vec{R}} = \vec{a} + 2\vec{\omega} \times \vec{V} + (\ddot{\vec{i}}X + \ddot{\vec{j}}Y + \ddot{\vec{k}}Z) \tag{7}$$

where $\vec{a}$ is the relative acceleration of point P as measured in the X, Y, Z system regarded as non-rotating. If equation (4) is differentiated, use being made of equation (6), there results

$$\ddot{\vec{R}} = \vec{a} + 2\vec{\omega} \times \vec{V} + \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times (\vec{\omega} \times \vec{R}) \tag{8}$$

If $\vec{D}$ is zero or small compared to $\vec{R}$, the last three terms in the right member of equation (8) represent the error that is made in determining the relative acceleration when the own-ship rotation is neglected. It may also be shown by differentiating equation (5) that the angular acceleration of the coordinate system as measured in the rotating coordinate system is the same vector as that measured in the X_o, Y_o, Z_o system:

$$\frac{d\vec{\omega}}{dt} = \vec{i}\dot{\omega}_X + \vec{j}\dot{\omega}_Y + \vec{k}\dot{\omega}_Z = \frac{\partial\vec{\omega}}{\partial t} \tag{9}$$

The effect upon the resulting trajectory of the error in relative acceleration due to the rotation of the reference coordinate system may be obtained without using equation (8). If the incorrect acceleration, $\vec{a} = \vec{i}\ddot{X} + \vec{j}\ddot{Y} + \vec{k}\ddot{Z}$, is integrated with respect to the time, use being made of the proper initial conditions, the result will be $\vec{V} = \vec{i}\dot{X} + \vec{j}\dot{Y} + \vec{k}\dot{Z}$.

The error made in measuring relative acceleration thus leads to the previously given errors in velocity and position.

Equations for the derivatives of $\vec{D}$ will now be developed. if $\vec{D} = \vec{I}_D D$, where $\vec{I}_D$ is a unit vector in the direction of $\vec{D}$, it will be seen from the previous discussion that

$$\dot{\vec{D}} = \vec{I}_D \dot{D} + D(\vec{\omega}_D \times \vec{I}_D) = \vec{I}_D \dot{D} + \vec{\omega}_D \times \vec{D}$$

$$\ddot{\vec{D}} = \vec{I}_D \ddot{D} + 2\dot{D}(\vec{\omega}_D \times \vec{I}_D) + \dot{\vec{\omega}}_D \times \vec{D} + \vec{\omega}_D \times (\vec{\omega}_D \times \vec{D})$$

where $\vec{\omega}_D$ is the angular velocity of $\vec{I}_D$ with respect to the X_o, Y_o, Z_o system. If there is no bending, $\dot{D} = \ddot{D} = 0$ and $\vec{\omega}_D$ can be determined from the equation

$$\vec{\omega}_D = \vec{I}_D \times (\vec{I}_D \times \vec{\Omega}_P)$$

where $\vec{\Omega}_P = \vec{I}_P \Omega_P$ is the angular velocity of the stabilized platform with respect to the vehicle. If there is bending, $\dot{D}$ and $\ddot{D}$ are not zero and are not necessarily negligible; moreover, $\vec{\omega}_D$ will either have to be measured directly or the value given by the above equation will have to be corrected by use of data from the instruments which measure the bending. Numerical differentiation will be needed if the instruments do not measure the correction and the derivatives directly.

The equations for the velocity and acceleration of point P as observed in the X_o, Y_o, Z_o system can now be written by use of equations (4) and (8):

$$\vec{V}_o = \dot{\vec{R}}_o = \dot{\vec{D}} + \vec{\omega} \times \vec{R} + \vec{V} \qquad (10)$$

$$\vec{a}_o = \ddot{\vec{R}}_o = \ddot{\vec{D}} + \vec{\omega} \times (\vec{\omega} \times \vec{R}) + \dot{\vec{\omega}} \times \vec{R} + 2\vec{\omega} \times \vec{V} + \vec{a}$$

where $\dot{\vec{D}}$ and $\ddot{\vec{D}}$ are given by the equations of the previous paragraph. If point P is motionless with respect to the X, Y, Z system, $\vec{V}$ and $\vec{a}$ are zero. The remaining terms in the right members of equations (10) are said to be the velocity of transport and the acceleration of transport of the X, Y, Z system with respect to the X_o, Y_o, Z_o system.

It is assumed in the foregoing discussion that the vector quantities

$\vec{V} = \vec{i}\dot{X} + \vec{j}\dot{Y} + \vec{k}\dot{Z}$ and $\vec{a} = \vec{i}\ddot{X} + \vec{j}\ddot{Y} + \vec{k}\ddot{Z}$ are determined without instrumentation error either by direct measurement or by numerical differentiation of the functions X(t), Y(t), and Z(t). Direct measurements could be made by use of electromagnetic radiation, for example.

The relative acceleration of one vehicle with respect to another could be determined indirectly by use of three inertial accelerometers in each vehicle; each set of accelerometers would be mounted on a stabilized platform with their sensitive axes mutually orthogonal. Similarly, the change in relative velocity as a function of time could be determined by use of three inertial velocimeters in each vehicle. In either case, the stabilized platforms should have known and preferably identical orientations with respect to an external reference; also the initial relative position and velocity of one vehicle with respect to the other would have to be determined by another method before the data could be integrated (numerically) to obtain the relative trajectory. In general it would also be necessary to know the gravitational attraction acting on each vehicle; but if only short-duration, short-range relative trajectories are of interest and if the initial conditions for the integration can be obtained during the intercept, this requirement can be waived without introducing significant errors into the computed trajectory. Some of these statements are fairly obvious, but others require some amplification.

An inertial accelerometer measures the force acting on a known mass suspended in such a way that it has one degree of translational freedom. Some of these instruments do not indicate acceleration but give a pulse for each incremental change in velocity $\Delta V = at$ so that a count of the pulses gives the change in velocity since the beginning of measurement; these instruments are velocimeters even though the absolute velocity is not determined unless the initial velocity happens to be zero. In general it will be necessary to measure V_o (and R_o as well) separately.

For determining how accelerometer readings are affected by movement of the coordinate system, it will be assumed for the moment that X_o, Y_o, Z_o is a true inertial system (motionless with respect to the stars) so that Newton's second law applies to accelerations measured with respect to that system. The force on a test mass m attached to the moving X_1, Y_1, Z_1 system with origin O_1 will then be given by

$$\vec{F} = m\vec{a}_o - m\vec{G}_1 = m(\ddot{\vec{D}}_1 + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{R}_{1m}) + \dot{\vec{\omega}}_1 \times \vec{R}_{1m})$$

$$- m\vec{G}_1 = m(\vec{a}_{T1} - \vec{G}_1)$$

where $\vec{\omega}_1$ is the angular velocity of the X_1, Y_1, Z_1 system with respect to inertial space, $\vec{R}_{1m}$ is the vector from origin O_1 to the test mass, $\ddot{\vec{D}}_1$ is the vector from O_o to O_1, $m\vec{G}_1$ is the gravitational attraction, and $\vec{a}_{T1}$ is the acceleration of transport.

An inertial accelerometer located at the point defined by $\vec{R}_{1m}$ measures the component of $\vec{a}_{T1}$ which is parallel to its sensitive axis, but it also detects and is biased by the gravitational field (Newtonian attraction) at that point. If the motion of the accelerometer (or of the vehicle containing the accelerometer) is unconstrained, the reaction of the accelerometer to the gravitational component of $\vec{a}_{T1}$ is equal and opposite to the bias produced by the gravitational field. Thus the inertial accelerometer does not measure the acceleration due to the gravitational field although it can be used to determine that acceleration if the correct value of $\vec{a}_T$ is known or can be computed or if the motion of the accelerometer is suitably constrained. If $\vec{G}$ is the gravitational field at the point of measurement, the vector determined from accelerometer readings will be $\vec{a}_a = \vec{a}_T - \vec{G}$.

It will be seen from the second of equations (10) that the acceleration of transport of a second moving system X_2, Y_2, Z_2 with origin O_2 can be related to the motion of the X_1, Y_1, Z_1 system as follows:

$$\vec{a}_{T2} = \ddot{\vec{D}}_1 + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{R}_{12}) + \dot{\vec{\omega}}_1 \times \vec{R}_{12} + 2\vec{\omega}_1 \times \vec{V}_{21} + \vec{a}_{21}$$

$$+ \vec{\omega}_{21} \times (\vec{\omega}_{21} \times \vec{R}_{2m}) + \dot{\vec{\omega}}_{21} \times \vec{R}_{2m} = \vec{a}_{a2} + \vec{G}_2$$

where $\vec{R}_{12}$ is the vector from origin O_1 to origin O_2, $\vec{R}_{2m}$ is the vector from O_2 to the point of measurement, $\vec{V}_{21}$ is the relative velocity of the two systems, $\vec{a}_{21}$ is the relative acceleration, and $\vec{\omega}_{21}$ is the angular velocity of the second system with respect to the first system. If the accelerometers in the two systems are mounted on stabilized platforms, the vectors $\vec{\omega}_1$, $\dot{\vec{\omega}}_1$, $\vec{\omega}_{21}$, $\dot{\vec{\omega}}_{21}$, $\vec{R}_{1m}$, and $\vec{R}_{2m}$ will all be small or zero; and terms containing $\vec{R}_{1m}$ and $\vec{R}_{2m}$ will certainly be negligible.

Thus

$$\vec{a}_{a2} - \vec{a}_{a1} = \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{R}_{12}) + \dot{\vec{\omega}}_1 \times \vec{R}_{12} + 2\vec{\omega}_1 \times \vec{V}_{21} + \vec{a}_{21} - (\vec{G}_2 - \vec{G}_1)$$

which is, except for the last term, the relative acceleration of the X_2, Y_2, Z_2 system with respect to the X_1, Y_1, Z_1 system corrected for rotation.

The quantity $(\vec{G}_2 - \vec{G}_1)$ is negligible for short-duration, short-range relative trajectories but cannot be neglected when the obtainable initial conditions require determination of the trajectory over extended periods of time by purely inertial means. The two coordinate systems are preferably parallel but obviously must be synchronized (i. e., the orientation of X_2, Y_2, Z_2 with respect to X_1, Y_1, Z_1 must be known for all times of interest) in order for the subtraction to be performed. Each coordinate system could be stabilized with one axis vertical and one axis pointing to geodetic north, an arrangement frequently preferred for navigation or guidance purposes, but the two systems would then have a small relative rotation $(\vec{\omega}_{21} \neq 0)$. Assuming perfect stabilization, the effects of this angular velocity and of the slight and varying misalignment of the two systems would be negligible for the short-range, short-duration relative trajectories of present interest. It would of course be preferable in general to have the two coordinate systems non-rotating.

Criteria for Neglecting Own-Ship Angular Motions

Under certain conditions, own-ship angular motions may be small enough, or the duration of the missile-target encounter may be short enough, for these motions to be neglected. Criteria for neglecting own-ship angular motions will be developed in the following paragraphs for the three scoring situations of interest in this study. It should be remembered that these criteria can also be used in data reduction. Provision should be made for correcting for own-ship angular motions unless it can be demonstrated that they will always be negligible.

Three future USAF scoring requirements are specified by the contract for consideration in this study. The relative trajectory of the missile with respect to the target is wanted in each case for all relative ranges $R \leq R_{Max}$. The first scoring problem is the trajectory of a missile with closing rates of 6000 to 10, 000 ft/sec at altitudes of 70, 000 to 200, 000 ft with R_{Max} = 3000 ft. The allowable position error is ± 100 ft; this is a relative accuracy of one part in thirty.

The error due to angular velocity alone cannot exceed the total allowable error minus the error from sources other than angular velocity. The maximum position error due to $\vec{\omega}$ is given by the expression developed in the preceding section. The maximum time duration of these trajectories is 6000 ft/6000 ft/sec = 1 sec. Then in order for angular velocity $\vec{\omega}$ of the reference system to introduce negligible trajectory error,

$$|R_{Max}\,\omega_{Max}\,\Delta t_{Max}| = 3000\ |\omega_{Max}| \leq 100\ \text{ft} - \left(\begin{matrix}\text{position error}\\ \text{not due to } \vec{\omega}\end{matrix}\right)$$

$$|\omega_{Max}| \leq 0.033\ \text{rad/sec} - (\text{position error not due to } \vec{\omega})/3000 \quad (11)$$

The fraction of the total allowable error to be allocated to angular-velocity effect is arbitrary; the choice will depend on the normal behavior of the scorer vehicle and on the size of the remaining error. Thus the error not due to $\vec{\omega}$ may be so large that no allowance can be made for uncorrected own-ship motions; also the normal angular rates of the vehicle may be so large that there is very little probability of their even being negligible. These remarks apply to any of the scoring situations being considered.

The angular velocity criterion expressed by equation (11) is conservative. A statistical approach could probably be justified for the most part, but the resulting possibly more liberal criterion would involve an arbitrary relation between the maximum permissible probable error due to neglecting $\vec{\omega}$ and the error bound developed in the previous section. Since the criteria given here are at the best preliminary estimates subject to modification by experience, this alternate approach is not attractive.

For satellite-vs-satellite encounters, scoring is required over the range interval of 0 to 500 ft; the maximum permissible position error is taken to be one part in thirty or about 15 ft. If a minimum closing rate of 2000 ft/sec is assumed, then a maximum time interval of 1000 ft/2000 ft/sec = 0.5 sec is of interest for such attacks. Then for the angular velocity $\vec{\omega}$ of the reference coordinate system to introduce negligible trajectory error,

$$|R_{Max}\,\omega_{Max}\,\Delta t_{Max}| \leq 15\ \text{ft} - (\text{position error not due to } \vec{\omega})$$

$$|\omega_{Max}| \leq 0.06\ \text{rad/sec} - (\text{position error not due to } \vec{\omega})/250 \quad (12)$$

For scoring attacks on ICBM's a range interval of interest of 0 to 2000 ft is specified. During all three phases of the ICBM

trajectory (boost, mid-course, and re-entry) the attacks probably would occur in the forward hemisphere of the ICBM, and thus large closing rates would be expected. The smallest closing rates would occur for attacks in which the weapon would be deployed in the path of the on-coming ICBM and would have comparatively little velocity of its own. The smallest closing rate could be expected to be at least of the order of 10, 000 ft/sec, so that a change in range of 4000 feet corresponds to an attack duration no greater than 0.4 sec. If the total position error is to be held to ± 100 ft, the bound on $\vec{\omega}$ would be

$$|\omega_{Max}| \leq 0.125 \text{ rad/sec - (position error not due to } \vec{\omega})/800 \quad (13)$$

Elimination of Own-Ship Angular Velocity Effects from the Data

When the own-ship angular velocity is too large to be neglected, as indicated by the simple criteria of the last section, it must in some way be taken into account. The simplest way, from the viewpoint of data reduction, is to stabilize the reference axis system. This could be done by stabilizing either the entire vehicle or a platform carrying the data-gathering instruments. Stabilization of the entire vehicle would not be applicable where it interferes with the normal operation of the vehicle in the performance of its mission. It would thus not be applicable to scoring encounters in the atmosphere or to attacks on an ICBM but might be applicable to satellite-vs-satellite encounters. Disturbing forces acting to change the orientation of the satellite would be quite small, and stabilization could be maintained with little expenditure of fuel and power. Further, such stabilization might be needed during attacks to minimize missile-release problems. Stabilization of the instrument platform alone might be indicated for attacks on instrumented ICBM's and for attacks within the atmosphere. Provision should be made for measuring bending if there is any likelihood that it will be excessive.

If stabilization were used, the permissible errors in the stabilized trajectory would be the same as those given in the preceding section for the case of no stabilization. The total change in the angular orientation of the stabilized platform would be no greater than the scalar sum of the angular changes about three mutually perpendicular axes. If the angular rate, Δp, about any axis did not exceed one third of the permissible value of $\vec{\omega}$ for the case of no stabilization, the trajectory errors due to these rates would not be excessive:

$$|\Delta p_{\text{permissible}}| \leq (1/3)\, |\vec{\omega}_{\text{permissible without stabilization}}|$$

where Δp includes the designed motion of the system with respect to inertial space (if any) as well as the drift rate. Since the drift rates

permitted by this criterion are likely to be well within the capabilities of available instrumentation, the selection of stabilization equipment might be dictated by considerations other than accuracy requirements, such as cost, weight, size, etc.

In many instances a stabilized platform can be used, but it will not be possible to mount all the instruments on the platform. In this case it will be necessary to measure and record the orientation of the platform with respect to a body-fixed coordinate system, and it may also be necessary to measure the bending angles for each data station. If relative velocities or accelerations are to be measured by non-inertial instruments, rate gyros or similar instruments could be used to measure the angular velocity components (at each data station unless it can be shown that bending will not contribute significantly to these rates). Alternately, $\vec{R}$ can be determined with respect to the body axes by numerical integration, and it will not be necessary to measure $\vec{\omega}$. The latter method is preferable unless $\vec{V}_o$ and/or $\vec{a}_o$ are wanted. In view of the criteria given in the last section, it appears that the angles should be recorded to the nearest milliradian; the other accuracy requirements are the same as those considered in the last paragraph.

It should be mentioned that some vehicles will use a stabilized system which has one axis "vertical" and one axis pointing either in the direction of travel or to geodetic north. The word vertical is used loosely here; an exact definition cannot be given without considering a specific application and a specific method of determining the vertical. A stabilized system of this type is useful in certain types of satellites and missiles and for navigation purposes in aircraft. The system will rotate slowly as the vehicle changes its position above the earth, but the rotation is negligible for present purposes.

An inertial (non-rotating) system is useful for deep-space vehicles, for certain types of satellites, and for ICBM's. The orientation of such a system with respect to the earth at the beginning of the scoring encounter can be computed with sufficient accuracy given the initial orientation, the initial location, and the location of the intercept.

The last method of correcting for angular motions is to use instruments to measure these motions (including bending) and then to take them into account in data reduction. A minimum of three angular rate gyros, sensing angular rates about each of three mutually perpendicular axes in the instrumented object, would be sufficient to establish the changes in orientation which occur during the course of an attack. The permissible errors for these rate gyros would be the same as those for the gyros used for stabilization, as given above. The accuracy requirements on this instrumentation are thus not critical. Rate gyros

are commonly used for measurements of this type, but other instruments are available. The angular orientation could also be obtained from photographs of the object, but it is believed that this procedure is not generally applicable and might prove troublesome in data reduction.

Rate gyros have long been used to determine orientation as a function of time in tests made on rockets and aircraft. Reports giving the details of this use are readily available.

Integrating the angular velocity components to obtain the changes in the orientation angles is a troublesome process if the time from initial conditions to the end of the scoring encounter is at all long. The problem of obtaining initial conditions will usually limit the application of this method to vehicles such as aircraft which characteristically have small motions about a nominal orientation which can normally be used as the initial condition without introducing excessive error. It can be shown that the angles obtained by simple integration of the angular rates do not have an inherent rotational order and as a result do not define unique orientations of the vehicle; the discrepancies will be small, however, if the angles are small. The discrepancies can theoretically be eliminated by using a simultaneous integration process employing appropriate coordinate transformations, but the supposed improvement may be lost as a result of roughness in the data or of errors stemming from the vastly increased complexity of the calculations. In general, the probability of obtaining good accuracy over a long period of integration is low. In short, the method can hardly be recommended except as a last resort.

APPENDIX 3

RATE CHARACTERISTICS OF LINEAR RELATIVE TRAJECTORIES

APPENDIX 3

RATE CHARACTERISTICS OF LINEAR RELATIVE TRAJECTORIES

Introduction

In problems of scoring missile-target encounters, the angular rates of one of the objects as viewed from the other may be important variables. If the measuring equipment is sensitive to the angular rates, for example, the errors of observation will be dependent upon these angular rates. For planning, it is desirable to know at least the angular rates and the ranges of variation of the angular coordinates in order to be certain that the scoring requirements are compatible with the allowable scanning rates and fields of view of the sensors. It may also be useful to have some information on the general behavior of the angles and angular rates as functions of the time. In this paper a brief investigation of such matters is presented without regard to the particular kind of equipment to be used for scoring.

Azimuth and Elevation Angular Rates

The angular velocity $\vec{\omega}$ of the relative range vector $\vec{R}$ is given by

$$\vec{\omega} = \frac{\vec{R} \times \dot{\vec{R}}}{R^2} = i\omega_x + j\omega_y + k\omega_z \tag{1}$$

where i, j, k are unit vectors in an X, Y, Z rectangular Cartesian coordinate system; $\vec{\omega}$ is normal to both $\vec{R}$ and $\dot{\vec{R}}$.

For linear relative trajectoris, $\vec{R}$ is given by

$$\vec{R} = \vec{R}_o + \dot{\vec{R}}_o t \tag{2}$$

Most relative trajectories of interest are well represented by this linear approximation. Substitution of equation (2) into equation (1) gives

$$\vec{\omega} \cong \frac{\vec{R}_o \times \dot{\vec{R}}_o}{R^2} = \vec{M}/R^2 \tag{3}$$

the vector $\vec{M}$ being a constant vector. Clearly $|\vec{\omega}|$ has its maximum when $|\vec{R}|$ is a minimum; at this point $\vec{R}$ and $\dot{\vec{R}}$ are orthogonal to each other, so that

$$|\vec{\omega}|_{Max} \cong \frac{\text{Relative velocity}}{\text{Miss distance}} \tag{4}$$

Finite values of $|\vec{\omega}|$ then result for all except direct hits.

Certain angular rates of $\vec{R}$ are given by the vectors $\vec{\dot{A}}$ and $\vec{\dot{E}}$, or azimuth and elevation angular rates, respectively (see Figure 1).

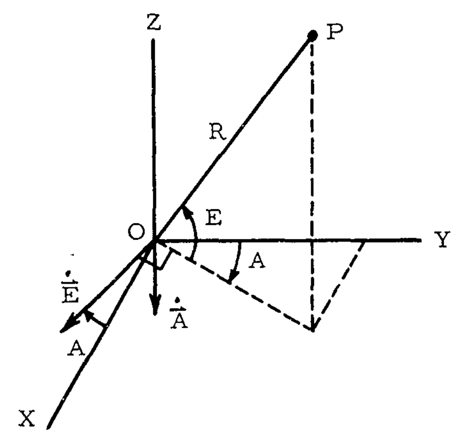

Figure 1. Azimuth and Elevation Angles

The vector $\vec{\dot{E}}$ is given by

$$\vec{\dot{E}} = \dot{E}(i \cos A - j \sin A) = (\omega_x \cos A - \omega_y \sin A)\vec{E}_1 \tag{5}$$

where $\vec{E}_1$ is a unit vector in the $\vec{\dot{E}}$ direction. The vector $\vec{\dot{A}}$ is given by

$$\vec{\dot{A}} = -k\dot{A} \tag{6}$$

which is not necessarily normal to $\vec{R}$. The vector $\vec{\dot{A}}$ will usually have a component, positive or negative, in the direction of $\vec{R}$.

It is desired to be able to put bounds on the angular rates $\dot{A}$ and $\dot{E}$, possibly by the use of $\vec{\omega}$. Since $\vec{\dot{A}}$ and $\vec{\dot{E}}$ are normal to each other, and the magnitude of their vector sum is greater than or equal to the length of $\vec{\omega}$, which is normal to $\vec{R}$,

$$\omega^2 \leq \dot{A}^2 + \dot{E}^2 \tag{7}$$

This puts a lower bound on the variables. Also since $\vec{\dot{E}}$ is normal to $\vec{R}$, and its length is equal to the projection of $\vec{\omega}$ in the direction of $\vec{\dot{E}}$,

$$|\vec{\omega}| \geq |\vec{\dot{E}}| \tag{8}$$

As $|\vec{\omega}|$ is bounded above by equation (4), this same bound serves as an upper bound for the magnitude of $\dot{\vec{E}}$. Unfortunately, $|\dot{A}|$ is not similarly bounded.

Putting bounds on $\dot{A}$ is somewhat more troublesome. One has

$$\tan A = X/Y$$

$$\dot{A} = (Y\dot{X} - X\dot{Y})/(X^2 + Y^2) \tag{9}$$

For linear relative trajectories, this reduces to

$$\dot{A} = K/(X^2 + Y^2) \tag{10}$$

which has its maximum magnitude when $(X^2 + Y^2)$ is a minimum. For short-duration trajectories, the maximum magnitude of the azimuth angular rate then occurs near the point where the relative coordinates satisfy $(X^2 + Y^2) = \text{minimum}$. Ordinarily, this point is different from the minimum-miss-distance point. Clearly

$$|\dot{A}|_{Max} \geq |K|/\text{miss distance}^2 \tag{11}$$

To go further with bounding $|\dot{A}|$, an expression for $\dot{A}$ in terms of $\vec{\omega}$ will be derived. One has the fact that the sum of the projections of $\dot{\vec{A}}$ and $\dot{\vec{E}}$ on $\vec{\omega}$ is equal to the length of $\vec{\omega}$, or

$$(\dot{\vec{A}} + \dot{\vec{E}}) \cdot \vec{\omega} = \omega^2 \tag{12}$$

Making use of equations (5) and (6) produces

$$-\dot{A}\omega_z + \dot{E}^2 = \omega^2$$

$$-\dot{A}\omega_z = \omega^2 - \dot{E}^2$$

$$-\dot{A} = \frac{\omega^2 - \dot{E}^2}{\omega_z} \tag{13}$$

Equation (13) is exact. Since $\omega^2 \geq \dot{E}^2$, one has

$$|\dot{A}| \leq \omega^2/|\omega_z| \tag{14}$$

From equation (1) one has for linear relative trajectories

$$\omega_z = (X_o \dot{Y}_o - Y_o \dot{X}_o)/R^2 = K/R^2 \tag{15}$$

For linear relative trajectories one may substitute from equation (15) into equation (14) to obtain

$$|\dot{A}| \leq \frac{R^2 \omega^2}{|X_o \dot{Y}_o - Y_o \dot{X}_o|} \leq \frac{(\text{Relative velocity})^2}{K} \tag{16}$$

the constant K being given by $K = |X_o \dot{Y}_o - Y_o \dot{X}_o| = |X\dot{Y} - Y\dot{X}|$.

Also, since $|\dot{\vec{E}}|$ is the projection of $\vec{\omega}$ along $\dot{\vec{E}}_1$, then $|\dot{\vec{E}}|$ is less than the length of the projection of $\vec{\omega}$ onto the XY plane, or

$$\dot{E}^2 \leq \omega_x^2 + \omega_y^2 \tag{17}$$

Substitution of equation (17) into equation (13) leads to

$$|\dot{A}| \geq |\omega_z| = K/R^2 \tag{18}$$

Another relation comparable to equation (13) may be derived by considering the differential displacements of $\vec{R}$ due to the angular motions. In Figure 2, a vector $\vec{R}$ undergoes small angular displacements, $\dot{A}\Delta t$ and $\dot{E}\Delta t$, causing a rotation of amount $\omega\Delta t$.

A vector $\vec{R}$, having originally the direction of OT in the X, Y, Z coordinate system of Figure 2, undergoes small angular displacements due to $\dot{A}$ and $\dot{E}$ acting over a differential time Δt, and is displaced to the direction of OQ. Consider the projections of $\vec{R}$ onto a sphere of radius S centered at O. Triangle TPQ is a plane right triangle up to first-order effects, so that

$$(\overline{TP})^2 + (\overline{PQ})^2 = (\overline{TQ})^2 \tag{19}$$

$$\left[(-\dot{A} \cos E)^2 + \dot{E}^2\right] (S\Delta t)^2 = \omega^2 (S\Delta t)^2$$

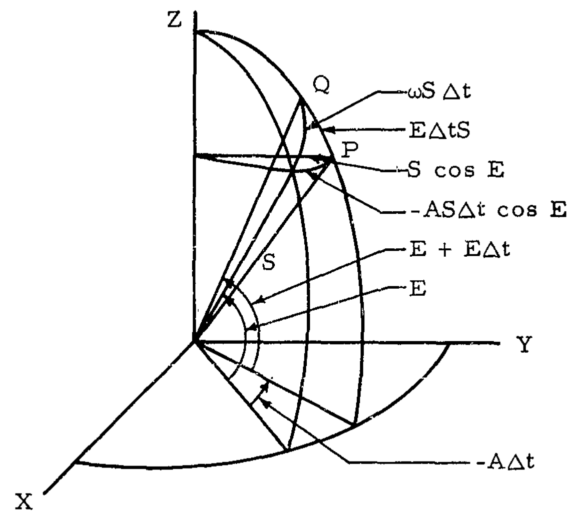

Figure 2. Infinitesimal Rotations

Hence

$$\dot{A}^2\cos^2 E + \dot{E}^2 = \omega^2 \tag{20}$$

On comparing equation (20) with equation (13), one sees that

$$\omega_z = -\dot{A}\cos^2 E \tag{21}$$

Equations (20) and (21) are applicable in the general case, They may also be derived by differentiation (use being made of relations for the relative rectangular coordinates in terms of the relative polar coordinates and equation (1)). Incidental to this process, one may obtain relations for ω_x and ω_y in terms of A, E, $\dot{A}$, and $\dot{E}$.

In view of equation (12) an immediate consequence of equation (20) is

$$\frac{|\omega_z|}{\cos E} \leq |\dot{A}| \leq \frac{|\omega|}{\cos E} \tag{22}$$

An Example of a Linear Relative Trajectory

An example of an attack with linear relative velocities is the following:

at $t = 0$, $\vec{R}_o = (50j + 2000k)$ ft

$$\dot{\vec{R}}_o = (40i - 160j - 4000k)\text{ft/sec}$$

For this example the relative rectangular coordinates, X, Y, Z, are given by

$$X = 40t$$
$$Y = 50 - 160t$$
$$Z = 2000 - 4000t$$

The azimuth and elevation angles are given by

$$\tan A = 40t/(50 - 160t)$$
$$\tan E = (2000 - 4000t)/(27,200t^2 - 16,000t + 2500)^{\frac{1}{2}}$$

The angular rate $\vec{\omega}$ of $\vec{R}$ is given by

$$\vec{\omega} = \frac{120,000i + 80,000j - 2000k}{4,002,500 - 16,016,000t + 16,027,200t^2}$$

The angular rate $\dot{A} = (Y\dot{X} - X\dot{Y})/(X^2 + Y^2)$

$$\dot{A} = 2000/(2500 - 16,000t + 27,200t^2)$$

The angular rate $\dot{E}$ is given by

$$\dot{E} = \left[\dot{Z}\sqrt{X^2 + Y^2} - Z\,\frac{(X\dot{X} + Y\dot{Y})}{\sqrt{X^2 + Y^2}}\right]/R^2$$

$$\dot{E} = \left[-4000\sqrt{2500 - 16,000t + 27,200t^2}\right.$$
$$\left. - (2000 - 4000t)\cdot\frac{(-8000 + 27200t)}{\sqrt{2500 - 16,000t + 27,200t^2}}\right]/(4,002,500$$
$$- 16,016,000t + 16,027,200t^2)$$

Also

$$R^2 = 4,002,500 - 16,016,000t + 16,027,200t^2$$

For a linear relative trajectory, the minimum range occurs at $t = -(\vec{R}_o \cdot \dot{\vec{R}}_o)/(\dot{\vec{R}}_o \cdot \dot{\vec{R}}_o)$, with

$$R^2_{Min} = R_o^2 - (\vec{R}_o \cdot \dot{\vec{R}}_o)^2/(\dot{\vec{R}}_o \cdot \dot{\vec{R}}_o)$$

These formulae may be obtained by differentiation of

$$R^2 = R_o^2 + 2\vec{R}_o \cdot \dot{\vec{R}}_o t + \dot{R}_o^2 t^2$$

In this example, $t_{R_{min}} \approx 0.49965$ and for this value of t, $R_{min} = 36.084$ feet.

Figure 3 is a plot of the relative rectangular coordinates X, Y, Z for this example. These coordinates are of course linear functions of time. Figure 4 is a plot of R vs t, which is linear except in the close neighborhood of the minimum range point. In this example $\vec{R}_o$ and $\dot{\vec{R}}$ are fairly closely aligned.

Figures 5 and 6 are plots of the azimuth angle A vs t and the elevation angle E vs t for this example. The angular rates $\dot{A}$ and $\dot{E}$ are shown in Figures 7 and 8. Note that for this example the maximum angular rate $\dot{A}$ does not occur near the point of closest approach, while $\dot{E}$ has its maximum at the point of closest approach.

The Angular Rates of $\vec{R}$ for a Certain System of Angular Coordinates

One system of angular coordinates measures the angles of the projection of R on the ZY and XZ planes (the angles α and β of Figure 9). The angles α and β are the ones measured if photo-potentiometers are used as sensors. They are related to the Cartesian X, Y, Z coordinates according to

$$\begin{aligned} R^2 &= X^2 + Y^2 + Z^2 \\ \tan\alpha &= Y/Z \\ \tan\beta &= X/Z \end{aligned} \tag{23}$$

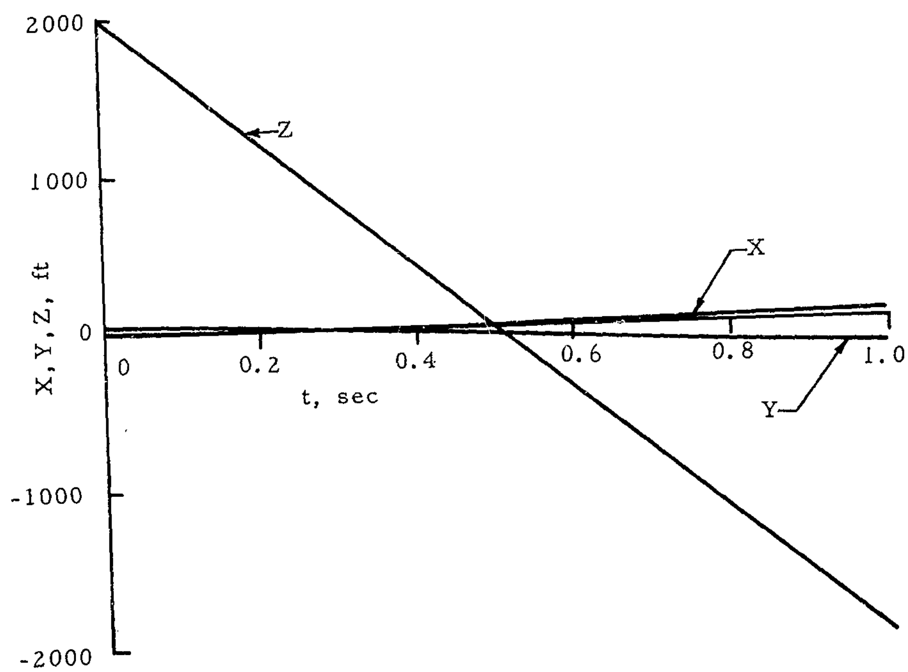

Figure 3

Example of Relative Coordinates X, Y, Z vs t

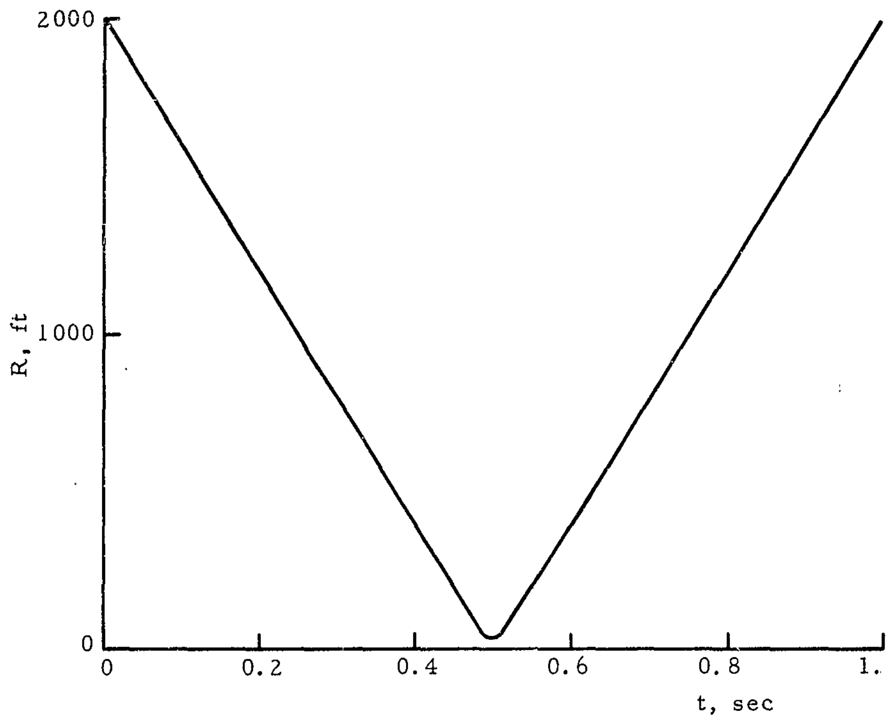

Figure 4

Example of Relative Range, R, vs t

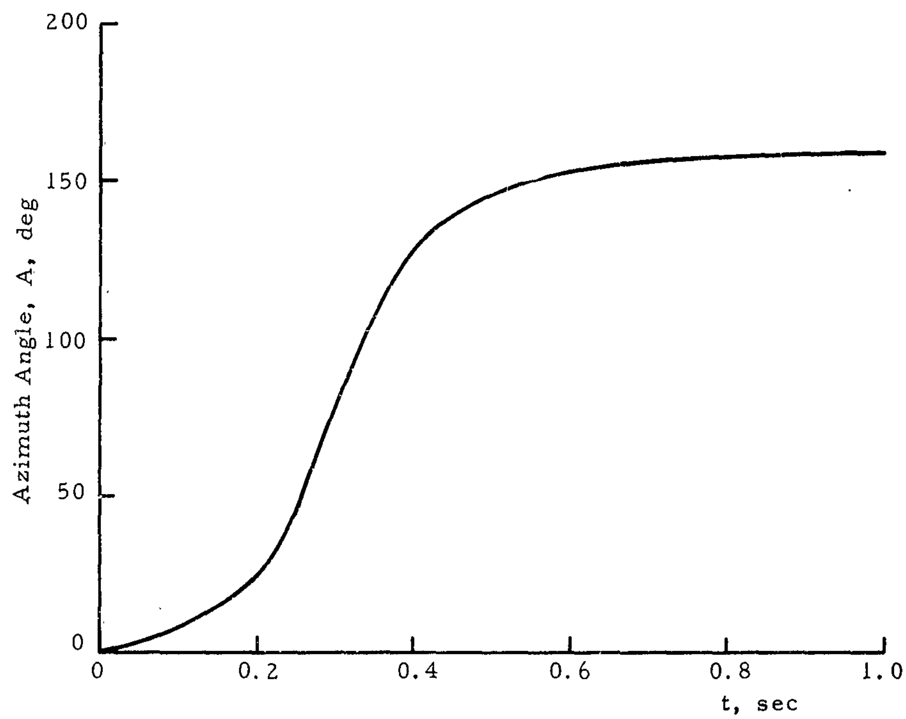

Figure 5

Example of Azimuth Angle, A vs t

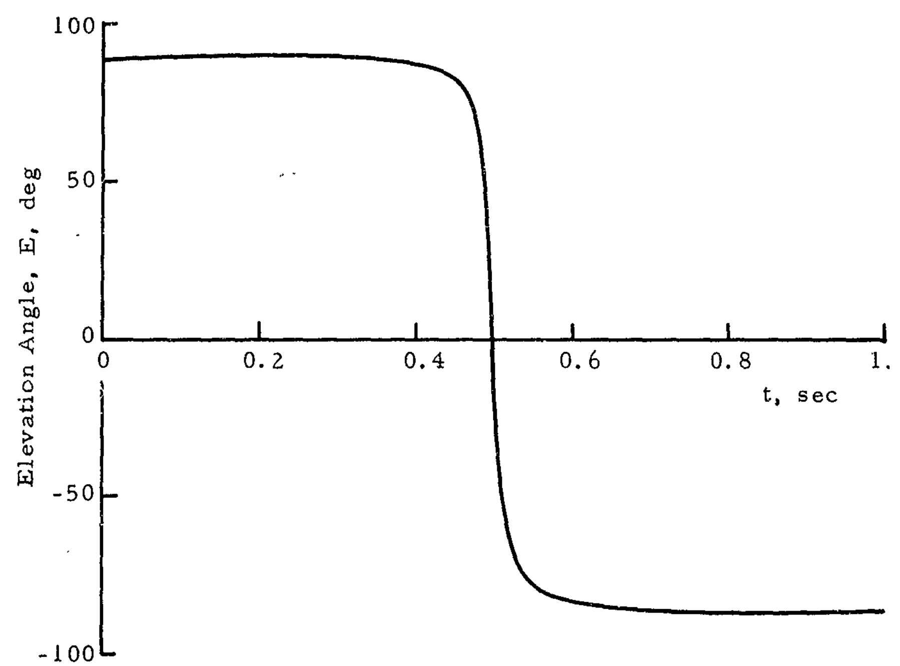

Figure 6

Example of Elevation Angle, E, vs t

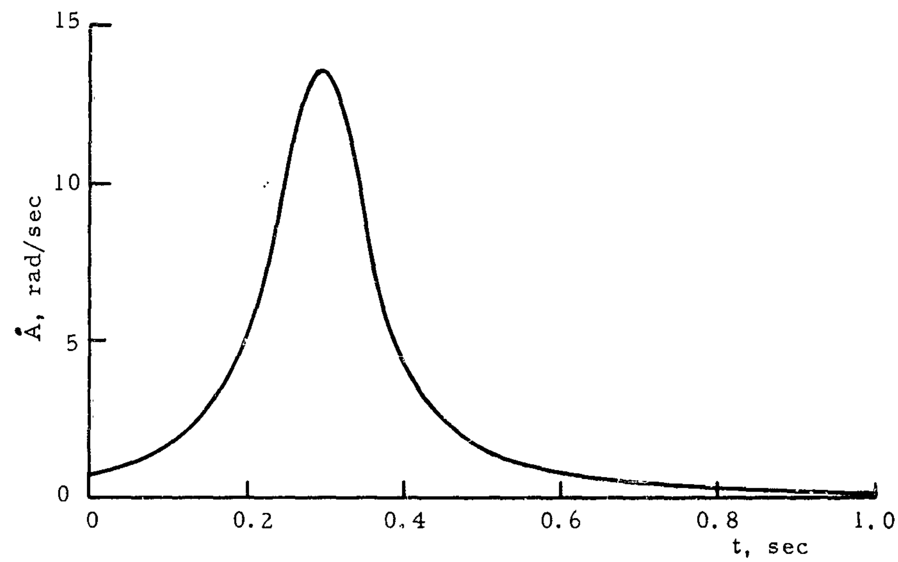

Figure 7
Example of Azimuth Angular Rate, $\dot{A}$, vs t

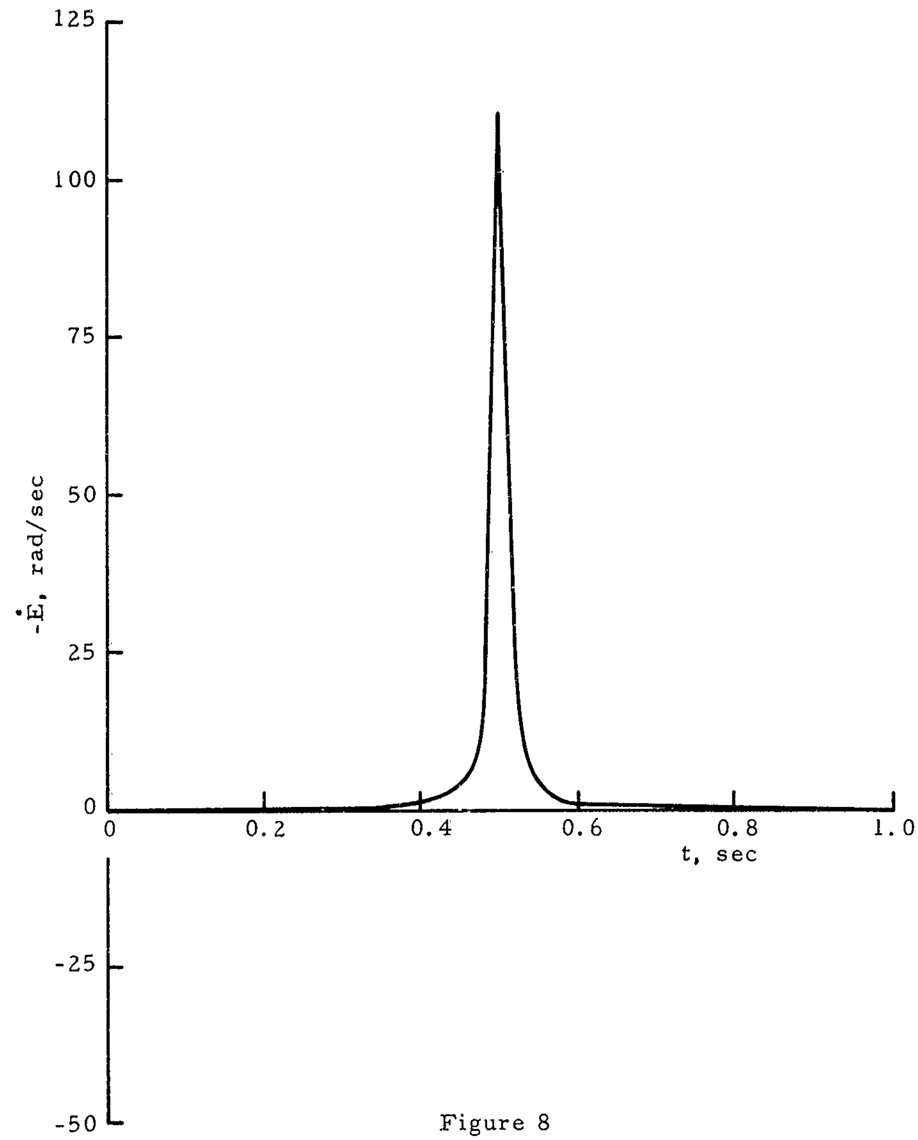

Figure 8
Example of Elevation Angular Rate, $\dot{E}$, vs t

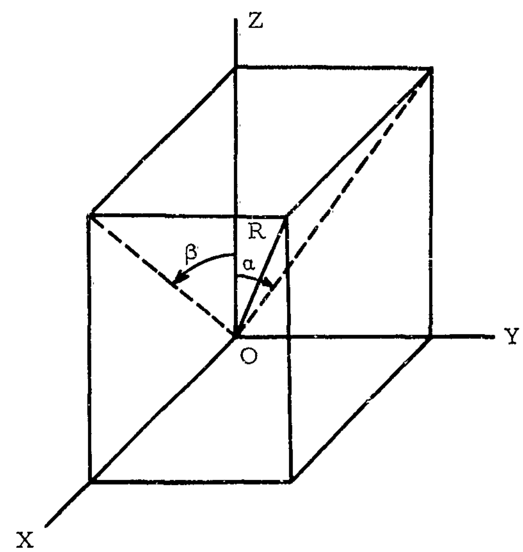

Figure 9. Angular Coordinates of R

This coordinate transformation has a double-valued inverse, but in application the values of Z are always taken positive, so that the inverse transform is given by

$$
\begin{aligned}
Z &= R/(1 + \tan^2\alpha + \tan^2\beta)^{\frac{1}{2}} \\
Y &= R \tan\alpha/(1 + \tan^2\alpha + \tan^2\beta)^{\frac{1}{2}} \qquad (24) \\
X &= R \tan\beta/(1 + \tan^2\alpha + \tan^2\beta)^{\frac{1}{2}}
\end{aligned}
$$

Differentiation gives

$$
\begin{aligned}
\dot{\alpha} &= -(Y\dot{Z} - Z\dot{Y})/(Y^2 + Z^2) \\
\dot{\beta} &= -(X\dot{Z} - Z\dot{X})/(X^2 + Z^2)
\end{aligned} \qquad (25)
$$

It is desired to put bounds on these rates in terms of appropriate variables of an attack. One has

$$\vec{\omega} = \frac{\vec{R} \times \dot{\vec{R}}}{R^2} = \frac{1}{R^2}\left[i(Y\dot{Z} - Z\dot{Y}) + j(Z\dot{X} - X\dot{Z}) + k(X\dot{Y} - Y\dot{X})\right]$$

Clearly, one has

$$|(Y\dot{Z} - Z\dot{Y})| \leq |\omega| R^2$$
$$|(X\dot{Z} - Z\dot{X})| \leq |\omega| R^2 \tag{26}$$

Also, one has

$$|\omega| \leq \text{Relative velocity}/R \leq \text{Relative velocity/Miss distance} \tag{27}$$

Then

$$|\dot{\alpha}| = |(Y\dot{Z} - Z\dot{Y})/(Y^2 + Z^2)| \leq \frac{|\omega|R^2}{Y^2 + Z^2}$$
$$|\dot{\beta}| = |X\dot{Z} - Z\dot{X}|/(X^2 + Z^2) \leq \frac{|\omega|R^2}{X^2 + Z^2} \tag{28}$$

For attacks with constant relative velocities all the quantities

$$|Y\dot{Z} - Z\dot{Y}|, \; |X\dot{Z} - Z\dot{X}|, \text{ and } |\omega|R^2$$

are constants. In particular, evaluating ω at the minimum range point,

$$\omega_{max} = \frac{\text{Relative velocity}}{\text{Miss distance}}, \quad |\omega|R^2 = (\text{Relative Velocity})(\text{Miss distance}) \tag{29}$$

and

$$|\dot{\alpha}| \leq \frac{(\text{Relative velocity})(\text{Miss distance})}{Y^2 + Z^2}$$
$$|\dot{\beta}| \leq \frac{(\text{Relative velocity})(\text{Miss distance})}{X^2 + Z^2} \tag{30}$$

Since $|\dot{\alpha}| = K_1/(Y^2 + Z^2)$, then $|\dot{\alpha}|$ has its maximum if $(Y^2 + Z^2)$ is minimum. Similarly $|\dot{\beta}|$ has its maximum if $(X^2 + Z^2)$ is minimum. These points in general are different from each other, and are in general different from the minimum range point.

Also, $|\dot{\alpha}|_{max} \geq K_1/(\text{Miss distance})^2$, with a similar inequality for $|\dot{\beta}|_{max}$. Closer bounds are given by

$$\begin{aligned} |\dot{\alpha}| &\geq K_1/R^2 \\ |\dot{\beta}| &\geq K_2/R^2 \end{aligned} \tag{31}$$

The components ω_x, ω_y, and ω_z of $\vec{\omega}$ may be expressed in terms of $\dot{\alpha}$, $\dot{\beta}$, α, and β by means of the coordinate relations (and their derivatives) given by equations (24) and (25), together with equation (1) for $\vec{\omega}$. However, no important or obviously useful information appears to be revealed by use of such a process. The bounds that have been given for the angular rates $\dot{\alpha}$ and $\dot{\beta}$ appear to be the most useful ones for planning.

APPENDIX 4

MINIMUM DATA REQUIREMENTS AND VARIANCE CONSIDERATIONS FOR ONE-, TWO-, AND THREE-STATION MEASUREMENTS

APPENDIX 4

MINIMUM DATA REQUIREMENTS AND VARIANCE CONSIDERATIONS FOR ONE-, TWO-, AND THREE-STATION MEASUREMENTS

Introduction

The determination of the position of an inaccessible point by triangulation is one of the oldest methods of indirect measurement. This method has received considerable theoretical attention in recent years, particularly in error analysis studies. Most of these theoretical treatments have considered angular measurements only and have been concerned with the over-determination of the point so that the errors in the indirectly measured quantities can be reduced or minimized by use of weighting factors or by making a least-squares adjustment. The study reported here was not concerned with the minimum sets of data required for one-, two-, and three-station measurement systems and was not restricted to angular measurements. The usual angle-only triangulation methods were of course included.

In rectangular coordinates, the minimum information is a set of values for the (X, Y, Z) coordinates of the point in question. In polar coordinates, the data required would be the distance from the origin to the point plus the azimuth and elevation angles of the line of sight (R, A, E). If measurements are taken from more than one position in the coordinate system, not all of the quantities (X, Y, Z), or equivalently (R, A, E), need be measured from each position. For example, an azimuth angle measured at one station, an azimuth and an elevation angle measured at a second station, and the distance between the two stations are sufficient to establish the position of a point P if the point does not fall on the extended line segment which connects the two measuring stations.

A few general remarks are made concerning one-station measurements, while all possible combinations of minimum sets of data are exhausted in a systematic manner for two-station measurements. Only a few selected cases of minimum data are considered for three-station measurements. The results are presented in an abbreviated form for easy reference. Equations and procedures are developed for specific cases to show how the variances (mean square errors) in the computed results are related to the variances in the measured data and how either the ranges of variation or the variances of the measured quantities are limited by the requirement that the variance in a given result should not exceed a specified value. A method is given for weighting the measured quantities so that the variance of a computed quantity can be minimized if redundant data are available.

One-Station Measurements

Figure 1 shows a point P and a single measuring station at point O which is taken as the origin of the X, Y, Z coordinate system. The position of point P can be determined by measuring its Cartesian coordinates (X, Y, Z), but it will usually be more convenient (or even necessary) to measure the spherical-polar coordinates (R, A, E). If either of these sets is specified, the other set is easily obtained through the simple relations

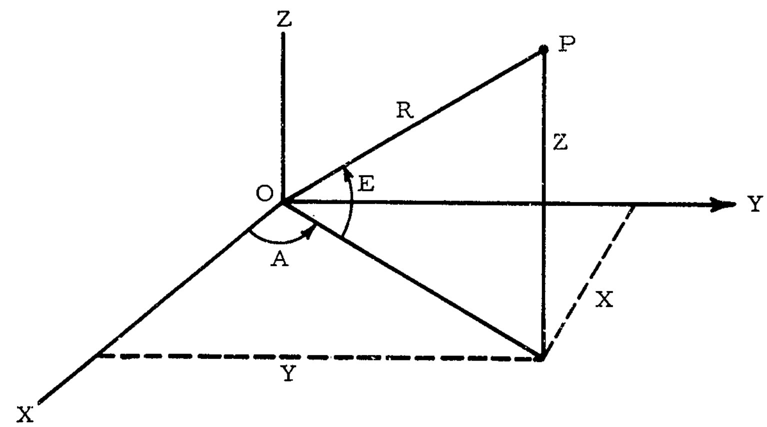

Figure 1

One-Station Measurements

$$X = R \cos E \cos A \qquad R = (X^2 + Y^2 + Z^2)^{\frac{1}{2}}$$

$$Y = R \cos E \sin A \qquad A = \tan^{-1}(Y/X)$$

$$Z = R \sin E \qquad E = \sin^{-1}(Z/R)$$

If the differentials are regarded as small errors or deviations from a correct set, the approximate errors in rectangular Cartesian coordinates and in spherical-polar coordinates are represented by the differentials as follows:

$$dX = dR \cos E \cos A + R[-\sin E \cos A dE - \cos E \sin A dA]$$

$$dY = dR \cos E \sin A + R[-\sin E \sin A dE + \cos E \cos A dA]$$

$$dZ = dR \sin E + R \cos E dE$$

and

$$dR = \frac{1}{(X^2 + Y^2 + Z^2)^{\frac{1}{2}}} [XdX + YdY + ZdZ]$$

$$dA = \frac{1}{(X^2 + Y^2)} [XdY - YdX]$$

$$dE = \frac{1}{(X^2 + Y^2)^{\frac{1}{2}}} \left[dZ - \frac{Z(XdX + YdY + ZdZ)}{(X^2 + Y^2 + Z^2)^{\frac{1}{2}}}\right]$$

No redundancy of data can result when only one-station measurements are used.

Two-Station Measurements

The geometry for two measuring stations will be as indicated in Figure 2.

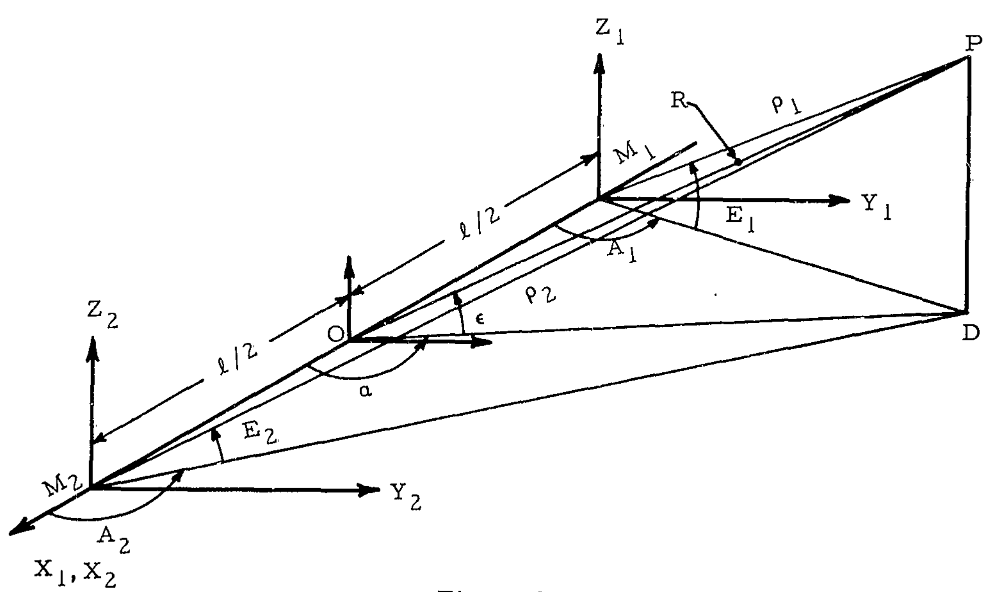

Figure 2

Two-Station Measurement System

The point O, located midway between the two measuring stations, M_1 and M_2, will be the point to which the position of the observed point, P, is to be related. The azimuth and elevation angles at point O will be designated as α and ϵ, respectively.

The separation of the two stations will be designated as ℓ. It will be assumed that this distance has been determined exactly, previous to the measurements made on the point P. Though this is not strictly true, the error in measuring ℓ should certainly be considerably less than the error in measuring ρ_1 or ρ_2. Following this same line of reasoning, it is assumed that point O is located at the exact midpoint of ℓ.

To determine all possible minimum sets of data from the two measuring stations, we observe that the number of combinations which can be made with six quantities when taken three at a time is $\frac{6!}{3!(6-3)!}$ or 20 different minimum sets. Of the 20 possible sets, some may be discarded since they do not afford sufficient information. As an example, if E_1, ρ_1, and A_2 are the three measured quantities being considered, the point P is not uniquely determined since there will be two points on the elevation cone of station M_1 which are located a distance ρ_1 from M_1 and fall in the plane established by A_2. This condition is pointed out in Figure 3. Both points P_1 and P_2 satisfy the same measured set of quantities E_1, ρ_1, and A_2.

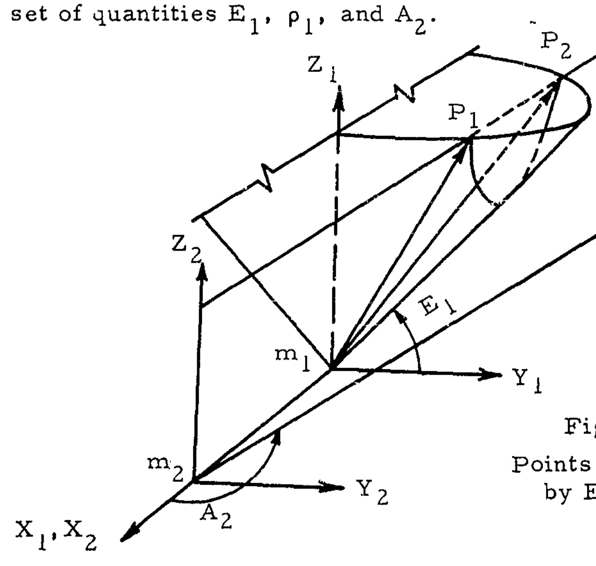

Figure 3
Points Determined by E_1, ρ_1, A_2

Because of the symmetry between the two measuring stations, some of the cases may be obtained as obvious extensions of others. An example of this symmetry effect is afforded by the two minimum data sets (A_1, E_1, A_2) and (A_2, E_2, A_1). The equations for the first set can be transformed to the equations for the second set, and vice versa, by interchanging subscripts 1 and 2.

Still other cases which may be discarded are those which turn out to be one-station sets of measured quantities. Only two such cases exist, namely (A_1, E_1, ρ_1) and (A_2, E_2, ρ_2).

The cases which will not be considered in detail, either because the point P is not uniquely determined or because they are one-station measurements, are as follows:

1. (A_1, E_2, ρ_1)
2. (A_1, E_2, ρ_2)
3. (A_2, E_1, ρ_2)
4. (A_2, E_1, ρ_1)
5. (E_1, E_2, ρ_1)
6. (E_1, E_2, ρ_2)
7. (E_1, ρ_1, ρ_2)
8. (E_2, ρ_1, ρ_2)
9. (A_1, E_1, ρ_1)
10. (A_2, E_2, ρ_2)

The equations for five out of the remaining ten cases will be given in the following paragraphs; the results for the other five cases are readily obtained from symmetry considerations. In these equations, angles α, A_1 and A_2 all have the same sign; the quadrant of α will have to be determined from the inequality $|A_2| \leq |\alpha| \leq |A_1|$. Note that only two out of the five cases are completely determinate; however, the indeterminate cases can sometimes be solved by restricting the fields of view of the measuring devices.

Case I. Measured Quantities: ℓ, A_1, E_1, A_2

$$R = \ell \left\{ \frac{G + H - J}{K} \right\}^{\frac{1}{2}}$$

$$\epsilon = \sin^{-1} \left\{ \frac{L}{[G + H - 4J]^{\frac{1}{2}}} \right\}$$

$$\alpha = \sin^{-1} \left\{ \frac{MN}{[G + HN^2 - 4J]^{\frac{1}{2}}} \right\}$$

where

$$G = \cos^2 E_1 \sin^2 (A_2 - A_1)$$

$$H = 4 \sin^2 A_2$$

$$J = \cos A_1 \sin A_2 \cos^2 E_1 \sin (A_2 - A_1)$$

$$K = 4G$$

$$L = 2 \sin A_2 \sin E_1$$

$$M = 2 \sin A_2 \sin A_1$$

$$N = \cos E_1$$

Case II. Measured Quantities: ℓ, A_1, E_1, E_2 (valid when $E_1 \geq E_2$)

$$R = \ell \left\{ \frac{G^2 JK}{H^2} + \frac{L}{4H} \right\}^{\frac{1}{2}}$$

$$\epsilon = \sin^{-1} \left\{ \frac{MGK}{[4JG^2K + HL]^{\frac{1}{2}}} \right\}$$

$$\alpha = \sin^{-1} \left\{ \tan A_1 \cdot \frac{2G^2K}{[4JG^2 + HL - M^2G^2K^2]^{\frac{1}{2}}} \right\}$$

where

$$G = \cos A_1 \cos E_1 \sin E_2$$

$$H = \sin^2 E_2 - \sin^2 E_1$$

$$J = \sin^2 E_2 + \sin^2 E_1$$

$$K = 1 + \sqrt{1 - \frac{H}{G^2}}$$

$$L = -3 \sin^2 E_2 - \sin^2 E_1$$

$$M = 2 \sin E_1 \sin E_2$$

Case III. Measured Quantities: ℓ, ρ_1, A_2, E_2

$$R = \left\{\rho_1^2 + \ell^2 G + \ell HJ\right\}^{\frac{1}{2}}$$

$$\epsilon = \cos^{-1}\left\{\frac{[J^2 \cos^2 E_2 + \ell^2 K + \ell HJL]^{\frac{1}{2}}}{[\rho_1^2 + \ell^2 G + \ell HJ]^{\frac{1}{2}}}\right\}$$

$$\alpha = \cos^{-1}\left\{\frac{[\rho_1^2 H^2 + \ell^2\{\frac{1}{4} - 2H^2(1 - H^2)\} + \ell HJ(2H^2 - 1)]^{\frac{1}{2}}}{[\rho_1^2 M + \ell^2\{\frac{1}{4} - M + H^2 L\} + \ell HJL]^{\frac{1}{2}}}\right\}$$

where ϵ is positive if E_2 is positive, and

$$G = (-\tfrac{3}{4} + \cos^2 A_2 \cos^2 E_2)$$

$$H = \cos A_2 \cos E_2$$

$$J = [\rho_1^2 - \ell^2(1 - \cos^2 A_2 \cos^2 E_2)]^{\frac{1}{2}}$$

$$K = (\tfrac{1}{4} - \sin^2 E_2 \cos^2 E_2 \cos^2 A_2)$$

$$L = \cos^2 E_2 - \sin^2 E_2$$

$$M = \cos^2 E_2$$

Case IV. Measured Quantities: ℓ, ρ_1, ρ_2, A_1

$$R = \frac{1}{2}\,(2G - \ell^2)^{\frac{1}{2}}$$

$$\epsilon = \cos^{-1}\left\{\frac{\ell\,[H^2 + J^2(H + \ell^2)^2]^{\frac{1}{2}}}{[2G - \ell^2]^{\frac{1}{2}}[\ell^2 - H]}\right\}$$

$$\alpha = \tan^{-1}\left\{\frac{J}{H}(H + \ell^2)\right\}$$

where the sign of ϵ is indeterminate, and

$$G = \rho_1^2 + \rho_2^2$$

$$H = \rho_1^2 - \rho_2^2$$

$$J = \tan A_1$$

Case V. Measured Quantities: ℓ, ρ_1, A_1, A_2

$$R = \left\{\rho_1^2 + \ell^2\left\{\frac{G - 4H}{4G}\right\}\right\}^{\frac{1}{2}}$$

$$\epsilon = \sin^{-1}\left\{\frac{2\,[\rho_1^2 G^2 - \ell^2 J^2]^{\frac{1}{2}}}{[4\rho_1^2 G^2 + \ell^2 G(G - 4H)]^{\frac{1}{2}}}\right\}$$

$$\alpha = \sin^{-1}\left\{\frac{2K}{[G^2 - 4(GH - J^2)]^{\frac{1}{2}}}\right\}$$

where the sign of ϵ is indeterminate, and

$$G = \sin(A_2 - A_1)$$

$$H = \cos A_1 \sin A_2$$

$$J = \sin A_2$$

$$K = \sin A_1 \sin A_2$$

Expressions for the differential errors in the computed quantities R, α, and ϵ resulting from small (differential) errors in the measured quantities ρ_1, ρ_2, A_1, A_2, E_1, and E_2 have been obtained by the straightforward method of taking differentials. While the method is relatively simple, the resulting expressions are exceedingly cumbersome and have not been included here. For possible future computational work and for reference, these differential expressions will be kept on file.

These differential error formulas may be used to set limits on the allowable errors of the measuring devices if the accuracy required for the computed quantities is specified. To facilitate this discussion of allowable errors, two-station geometry will be considered in which the azimuth angle is measured at one station and both the azimuth and elevation angles are measured at the other station; this is one of the two completely determinate cases considered above. For simplicity, the computed distance, R, is related to the station at which both azimuth and elevation angles are measured. The expression for R in terms of the measured angles can be written as follows:

$$\frac{R}{\ell} = \frac{\sin A_2}{\cos E_1 \sin(A_2 - A_1)} = f(A_1, A_2, E_1)$$

where ℓ is the station separation. The variance (or average square of the error) in R is defined by

$$\sigma_R^2 = \underset{M \to \infty}{\text{Limit}} \frac{1}{M} \sum_{i=1}^{M} (dR)_i^2 = \sum_{n=1}^{N} \left(\frac{\partial R}{\partial X_n}\right)^2 \sigma_{X_n}^2$$

where $R = R(X_1, X_2, \ldots, X_N)$ and $\sigma_{X_n}^2$ is the variance in variable X_n.

The result obtained for this case can be written as follows:

$$\sigma_R^2 = \ell^2 \left\{ K_{A_1}^2 \sigma_{A_1}^2 + K_{A_2}^2 \sigma_{A_2}^2 + K_{E_1}^2 \sigma_{E_1}^2 \right\}$$

where

$$K_{A_1}^2 = \left[\frac{\sin A_2 \cos(A_2 - A_1)}{\cos E_1 \sin^2(A_2 - A_1)}\right]^2$$

$$K_{A_2}^2 = \left[\frac{\sin(A_2 - A_1)\cos A_2 - \sin A_2 \cos(A_2 - A_1)}{\cos E_1 \sin^2(A_2 - A_1)}\right]^2$$

$$K_{E_1}^2 = \left[\frac{\sin A_2 \sin E_1}{\cos^2 E_1 \sin(A_2 - A_1)}\right]^2$$

For convenience, it is now assumed that the variances in the angular measurements are all equal; i.e.,

$$\sigma_{A_1}^2 = \sigma_{A_2}^2 = \sigma_{E_1}^2 \equiv \sigma^2$$

and the expression for the variance in R may then be rearranged to give

$$\left(\frac{\sigma_R}{\sigma R}\right)^2 = \frac{1}{f^2(A_1, A_2, E_1)} \cdot \left\{K_{A_1}^2 + K_{A_2}^2 + K_{E_1}^2\right\}$$

The right-hand side of this expression is dimensionless and plots of this expression and of the expression for $(\ell/R)^2$ may be obtained as functions of the angles A_1, E_1, and A_2. Such plots are given in Figures 4 through 15 which follow. The parameter A_2 is used as the connecting variable between the two sets of curves.

For showing how these curves may be interpreted, we consider the following design problem. Suppose the variance in R is required to be less than σ_{Max}^2 over a range of values for R, say $R \leq R_{Max}$. The fields of view of the angle-measuring devices and the variances σ^2 (for all angles measured by these devices) are specified. The question to be asked, then, is how long must the base line be for all these requirements on our detecting system to be fulfilled--assuming that they can be fulfilled. The first step is to determine values for $A_{2_{Min}}$ by using the curves of $\left(\frac{\sigma_R}{\sigma R}\right)^2$ vs A_2 for all $E_1 \leq E_{1_{Max}}$ and the given values of σ_{Max}^2, σ^2 (all angles), $A_{1_{Min}}$ and R_{Max}. Some of the values of $A_{2_{Min}}$ that would otherwise be acceptable may not be within

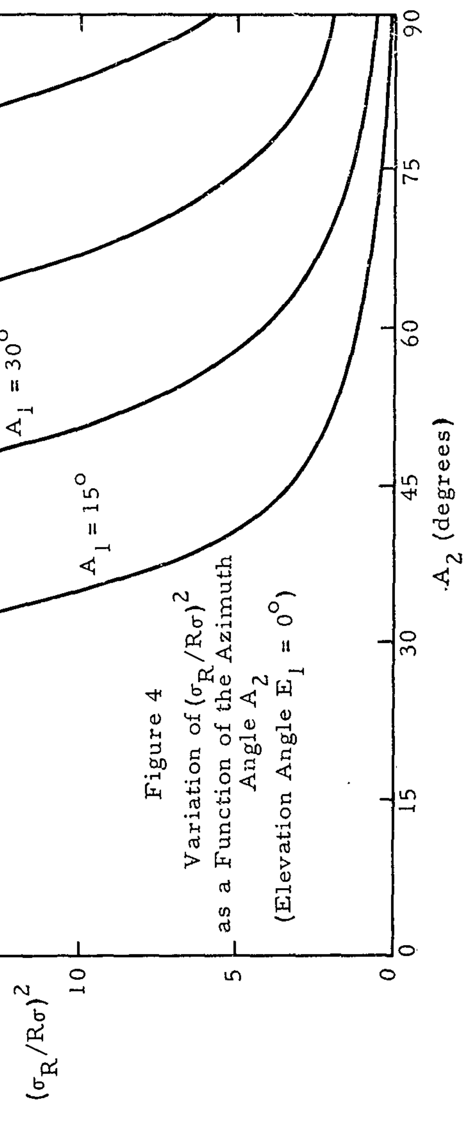

Figure 4
Variation of $(\sigma_R/R\sigma)^2$ as a Function of the Azimuth Angle A_2
(Elevation Angle $E_1 = 0^\circ$)

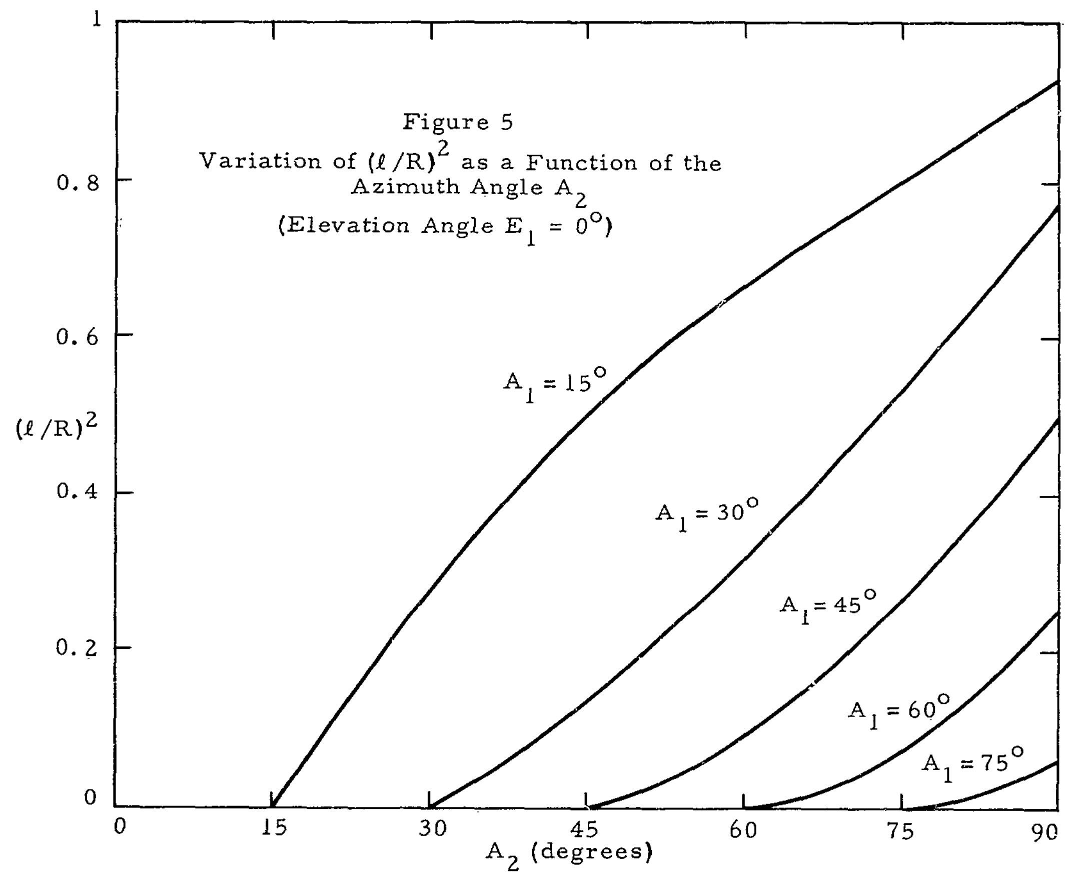

Figure 5
Variation of $(\ell/R)^2$ as a Function of the Azimuth Angle A_2
(Elevation Angle $E_1 = 0^\circ$)

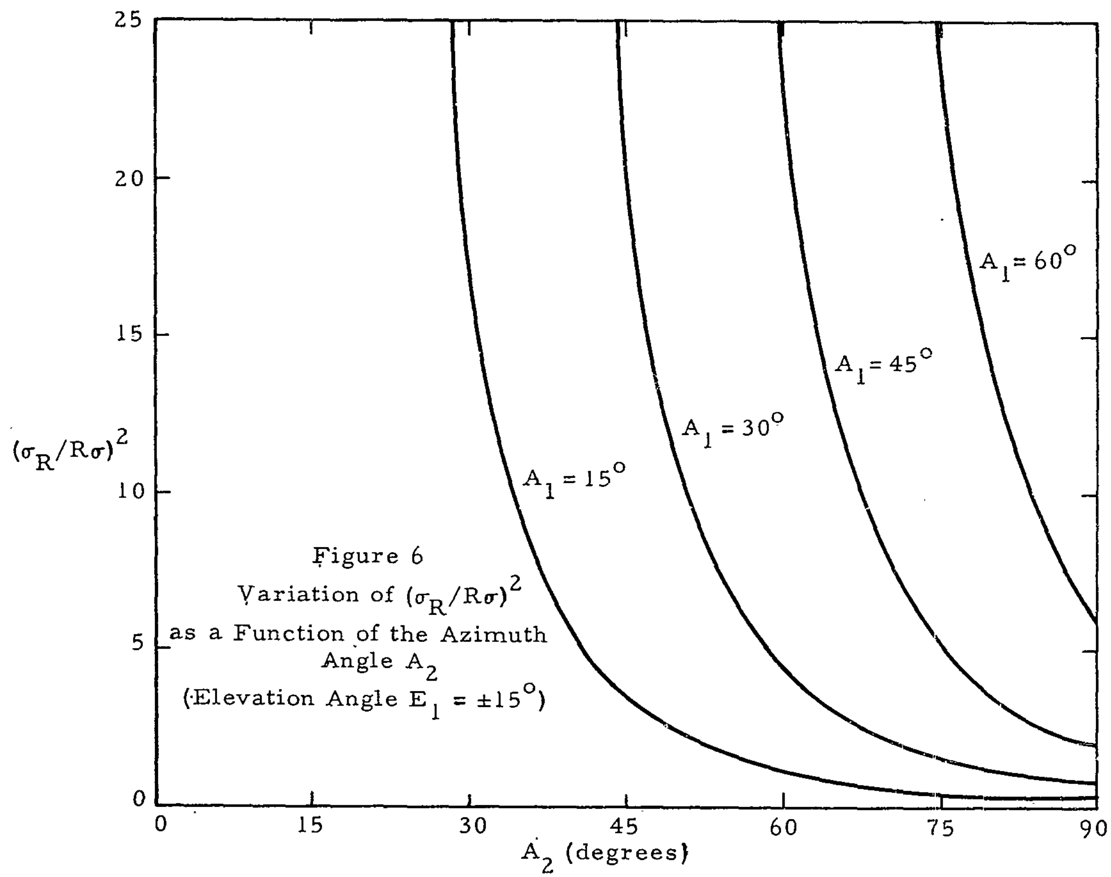

Figure 6
Variation of $(\sigma_R/R\sigma)^2$
as a Function of the Azimuth
Angle A_2
(Elevation Angle $E_1 = \pm 15^o$)

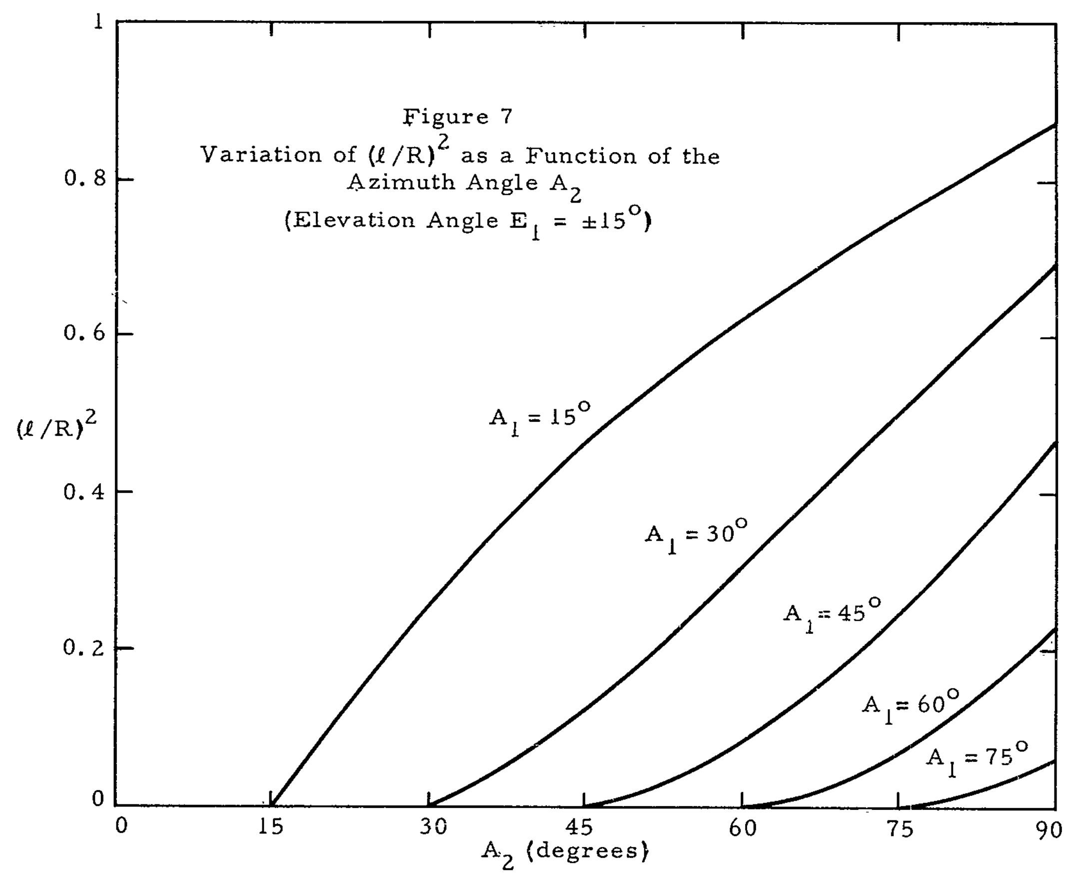

Figure 7
Variation of $(\ell/R)^2$ as a Function of the Azimuth Angle A_2
(Elevation Angle $E_1 = \pm 15^\circ$)

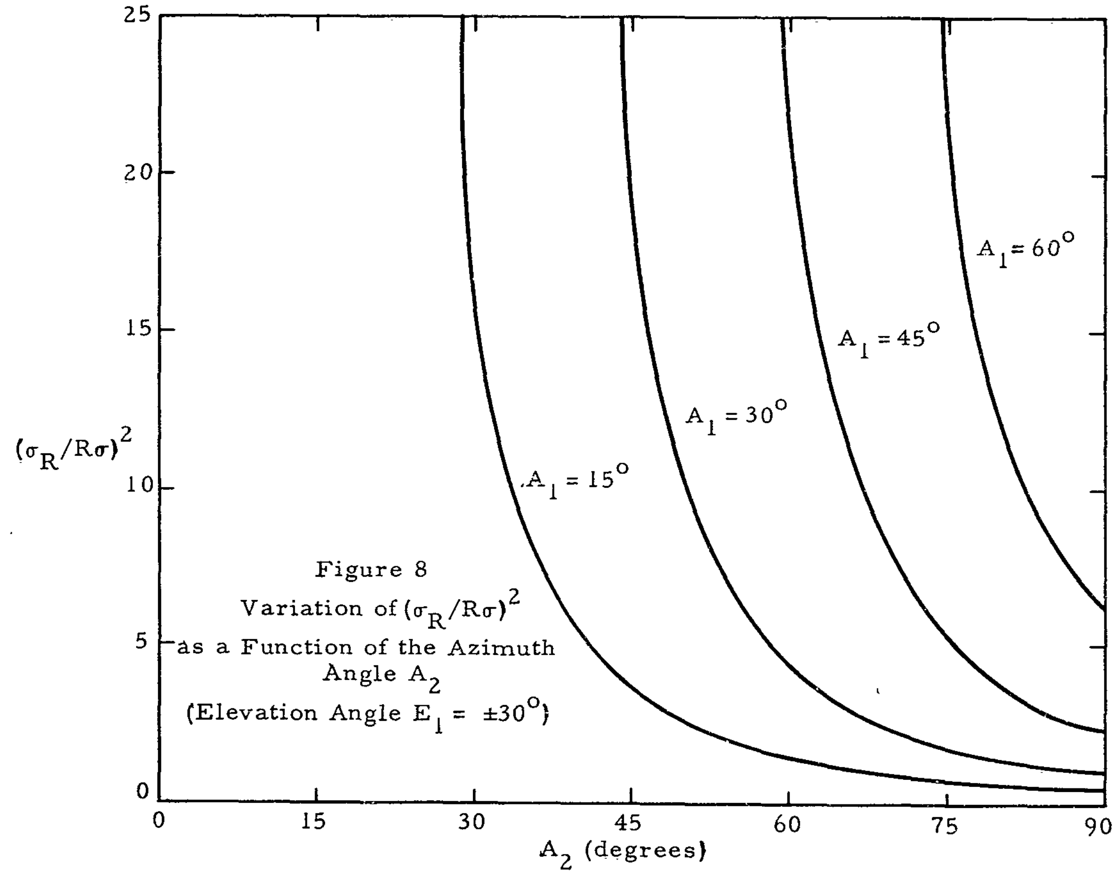

Figure 8
Variation of $(\sigma_R/R\sigma)^2$ as a Function of the Azimuth Angle A_2
(Elevation Angle $E_1 = \pm 30^\circ$)

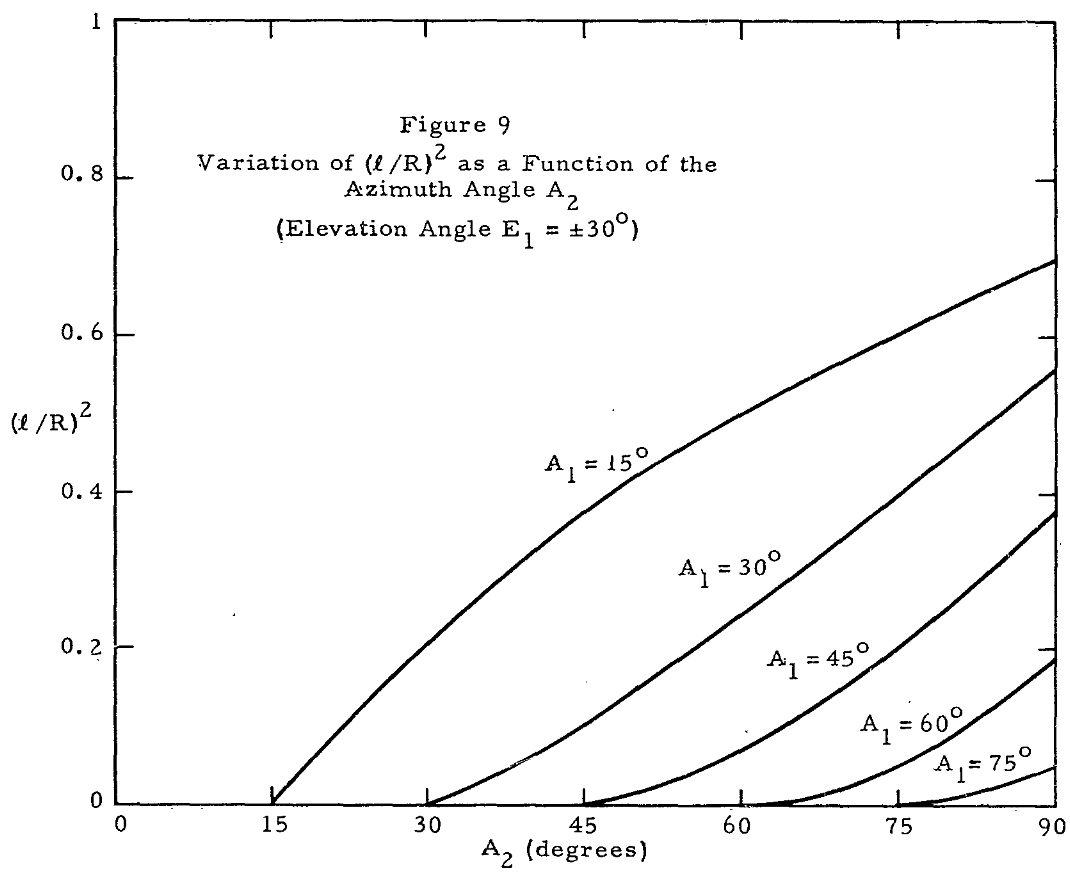

Figure 9
Variation of $(\ell/R)^2$ as a Function of the Azimuth Angle A_2
(Elevation Angle $E_1 = \pm 30^\circ$)

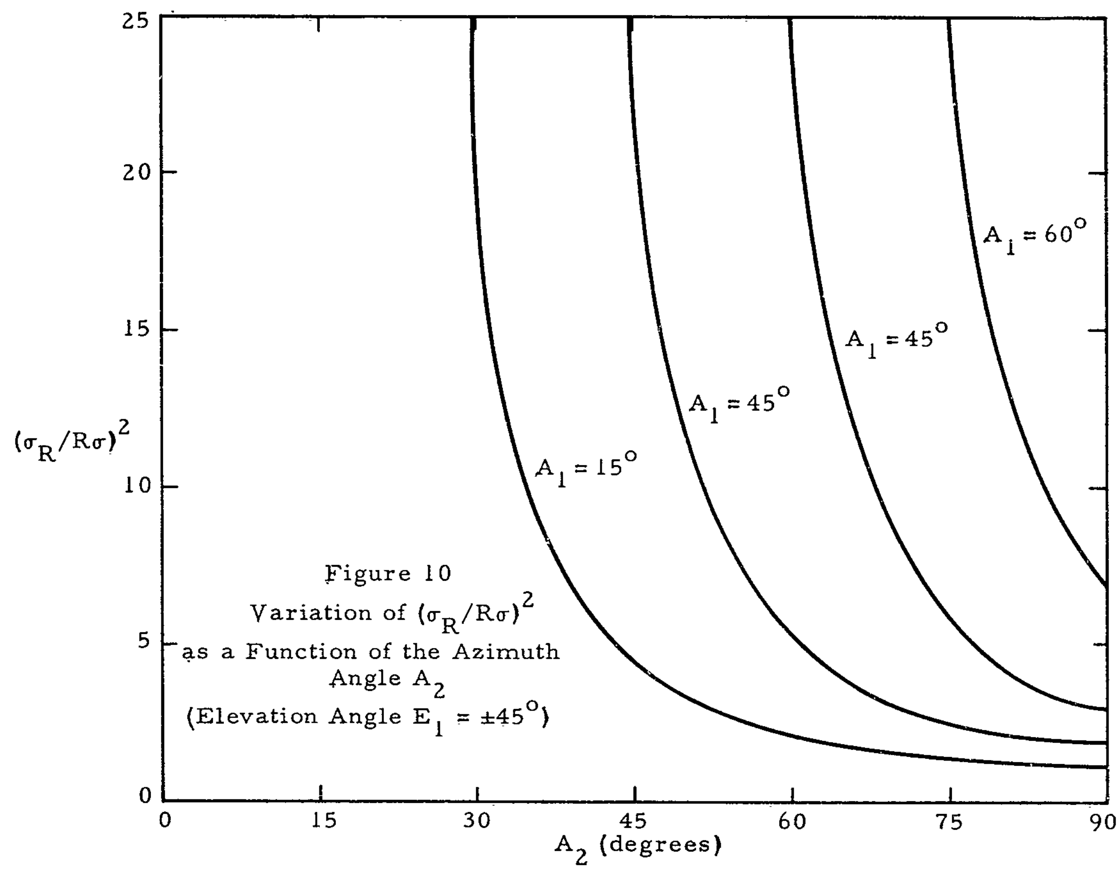

Figure 10
Variation of $(\sigma_R/R\sigma)^2$
as a Function of the Azimuth
Angle A_2
(Elevation Angle $E_1 = \pm 45^o$)

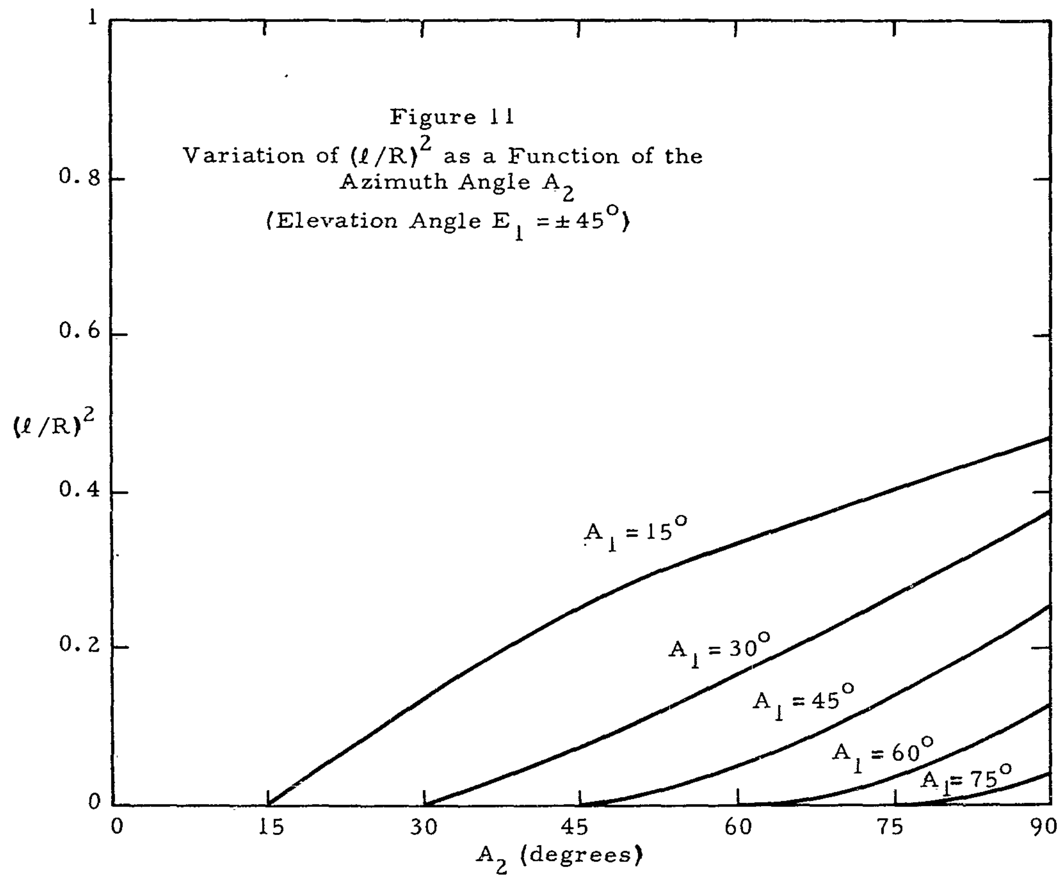

Figure 11
Variation of $(\ell/R)^2$ as a Function of the Azimuth Angle A_2
(Elevation Angle $E_1 = \pm 45°$)

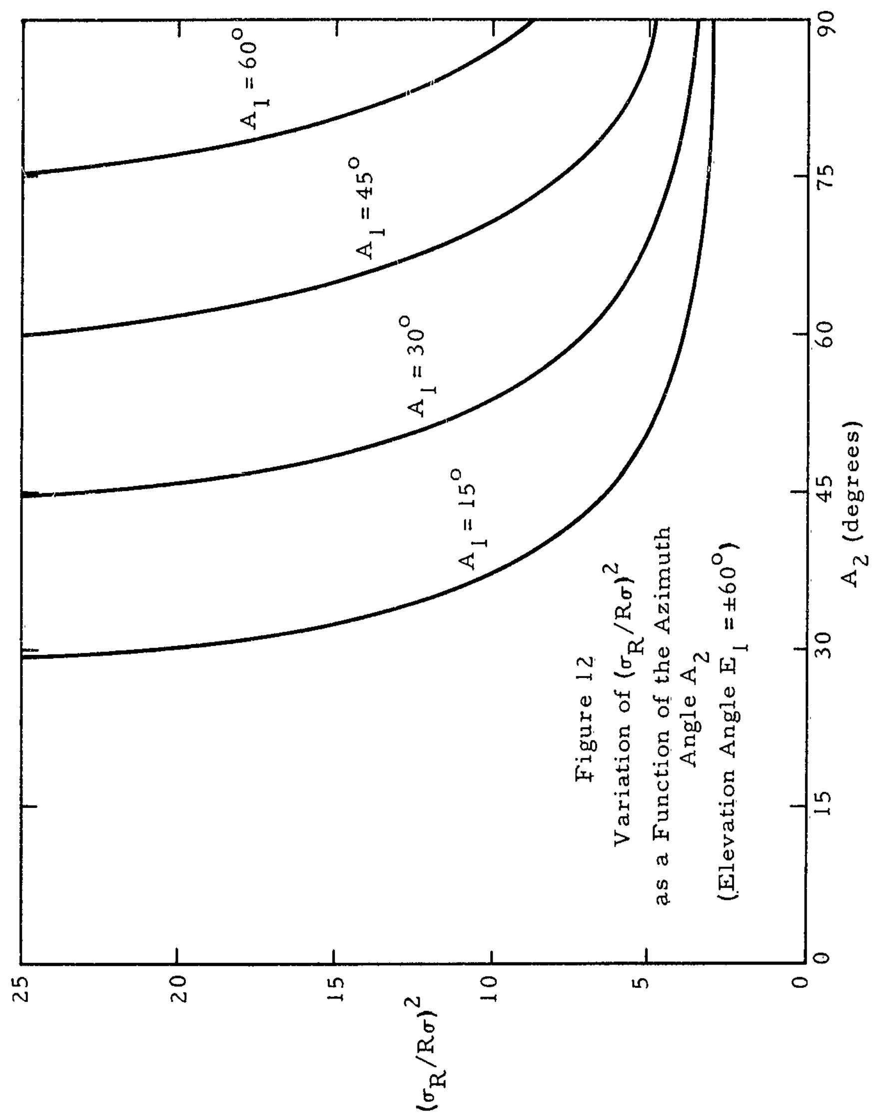

Figure 12
Variation of $(\sigma_R/R\sigma)^2$
as a Function of the Azimuth
Angle A_2
(Elevation Angle $E_1 = \pm 60^\circ$)

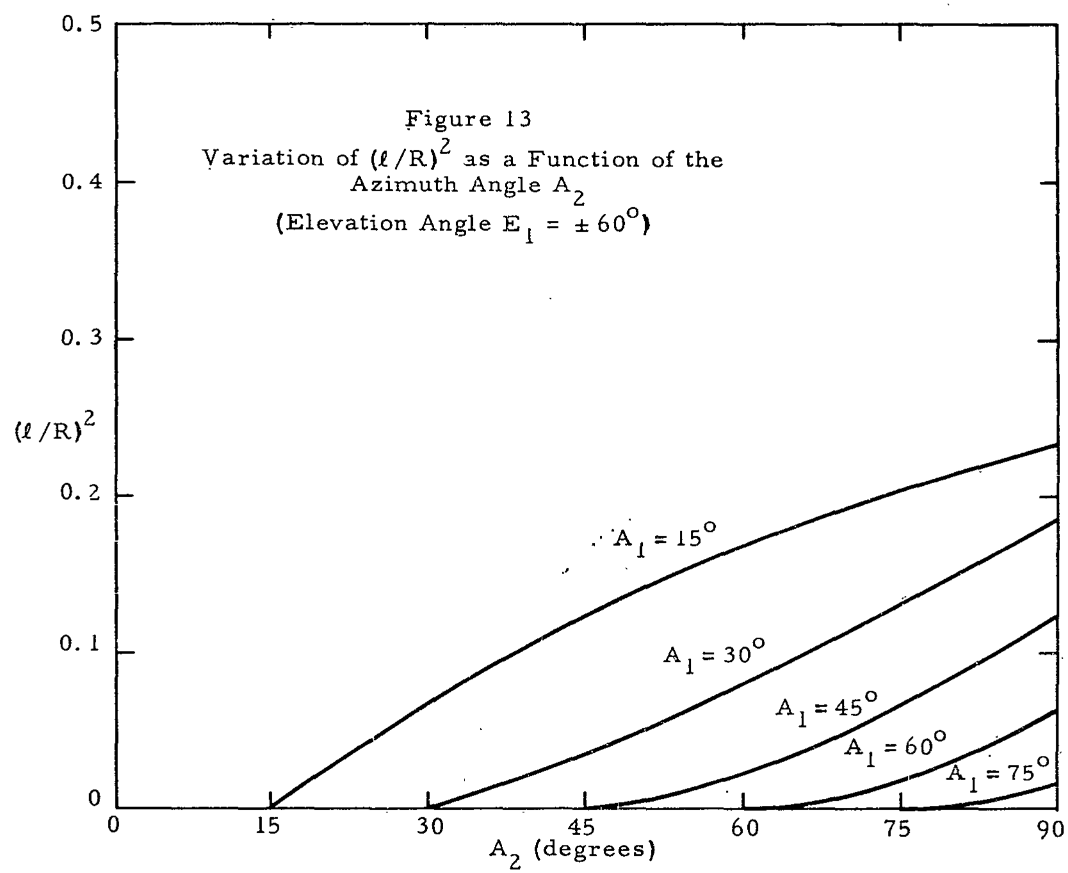

Figure 13

Variation of $(\ell/R)^2$ as a Function of the Azimuth Angle A_2

(Elevation Angle $E_1 = \pm 60^\circ$)

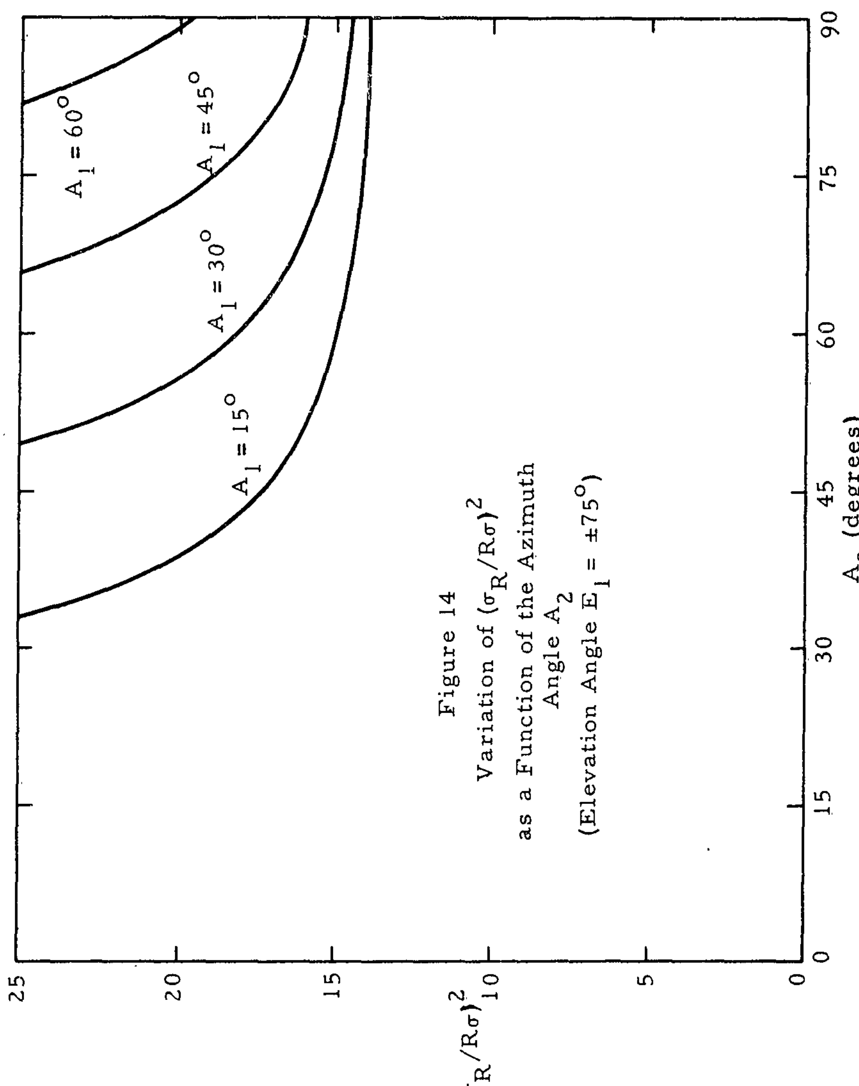

Figure 14
Variation of $(\sigma_R/R\sigma)^2$
as a Function of the Azimuth
Angle A_2
(Elevation Angle $E_1 = \pm 75^\circ$)

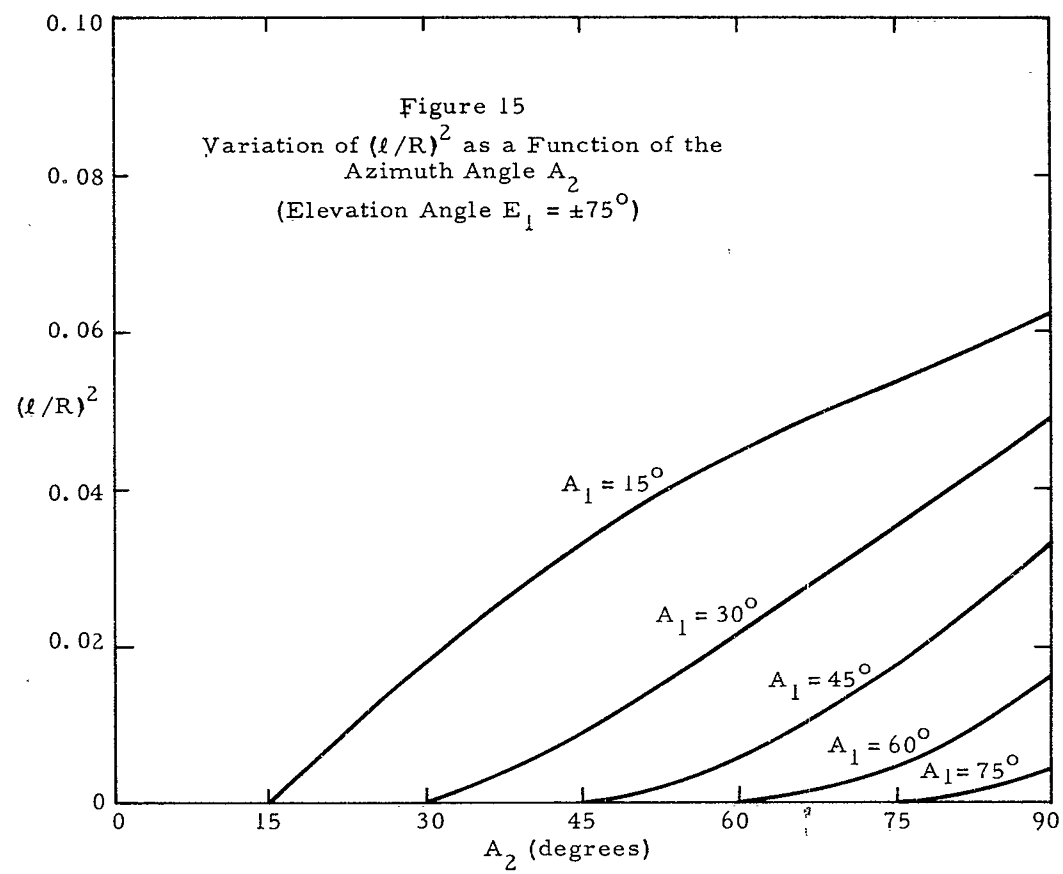

Figure 15
Variation of $(\ell/R)^2$ as a Function of the Azimuth Angle A_2
(Elevation Angle $E_1 = \pm 75^\circ$)

the field of view of the device; in such cases the field limit should be used. The values of $(\ell/R)^2$ corresponding to each set $(A_{1_{Min}}, A_{2_{Min}}, E_1)$ can now be read from the $(\ell/R)^2$ -vs-A_2 curves. Since R_{Max} is known, the required value of ℓ may be readily computed from the largest value of $(\ell/R)^2$ determined in this manner.

Two-Station Angle-Only Measurements--One Angle at Each Station

It is desired to investigate the geometrical and error relations for ranging with two-station measurements of angles only. In Figure 16, the observed object is at C, the base line AB is of known length, D, and angles A and B are determined from the azimuth and elevation angles measured at A and B by the relations $\cos A = \cos A_A \cos A_E$ and $\cos B = \cos B_A \cos B_E$ where A_A and A_E are the azimuth and elevation angles measured at A, etc.

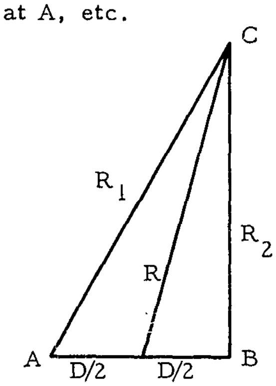

Figure 16
Two-Station Scoring Geometry

From the geometry,

$$R_1/D = \sin B/\sin(A + B)$$

$$R_2/D = \sin A/\sin(A + B)$$

$$\frac{R}{D} = \sqrt{\left(\frac{R_1^2 + R_2^2}{D^2} - \frac{1}{2}\right)\frac{1}{2}} = \sqrt{\left(\frac{\sin^2 A + \sin^2 B}{\sin^2(A+B)} - \frac{1}{2}\right)\frac{1}{2}}$$

The last equation was used to compute R/D as a function of (A + B) for B held fixed at each of several values: 15°, 30°, 60°, 90°. The results are plotted in Figures 17 through 20. It was noted that larger values of R/D occur for (A + B) near 180°, a condition that would exist when long ranges are involved. Additional computations were made to cover the cases with a subtended angle C on the interval $0\text{-}5^{\circ}$, and Figures 21 through 24 show plots of R/D vs B for the subtended angle C held fast. These plots show that for a constant angle C, maximum range occurs when R is normal to the base line, AB. Also, R/D is a highly non-linear function of C as C approaches zero.

The relations connecting the angular errors made in measuring the angles A and B, the errors of R, and the length, D, of the base line AB are of interest. Plots 17 through 24 show that the critical situations for accuracy are those with either A or B small and the other large, so that the subtended angle C is small. In such situations, R/D is large, and

$$R/D \sim R_1/D \sim R_2/D$$

Making use of

$$R/D \sim R_1/D = \sin B/\sin(A + B),$$

and taking differentials, ignoring for the present any errors in D, gives

$$\frac{dR}{D} \sim \frac{\cos B\, dB}{\sin(A + B)} - \frac{\sin B \cos(A + B)(dA + dB)}{\sin^2(A + B)}$$

Upon setting the measurement errors in A and B equal to each other, dA = dB, and multiplying dR/D by D/RdA, there results

$$\frac{dR}{RdA} = \frac{D}{R}\left[\frac{\cos B}{\sin(A + B)} - \frac{2 \sin B \cos(A + B)}{\sin^2(A + B)}\right] = \frac{D}{R} H(A, B)$$

Plots of H(A,B) vs (A + B) for B held fast were made (Figures 25 through 27). This function becomes large without limit for $(A + B) = 180^{\circ}$. If $B \leq 90^{\circ}$, and $(A + B) \geq 90^{\circ}$, the two terms in H(A,B) have the same algebraic sign for dA = dB. As (A + B) tends toward 180°, the second term predominates, so that

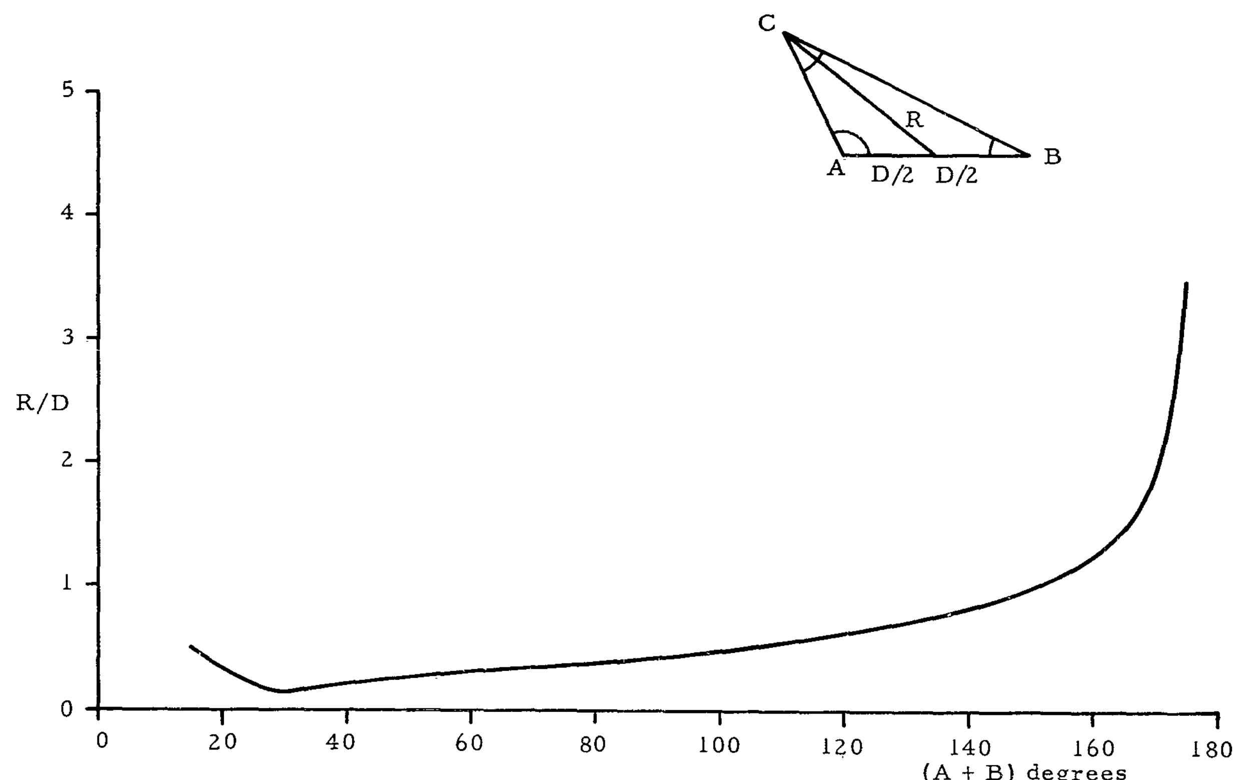

Figure 17

R/D vs (A+B) for B = 15°

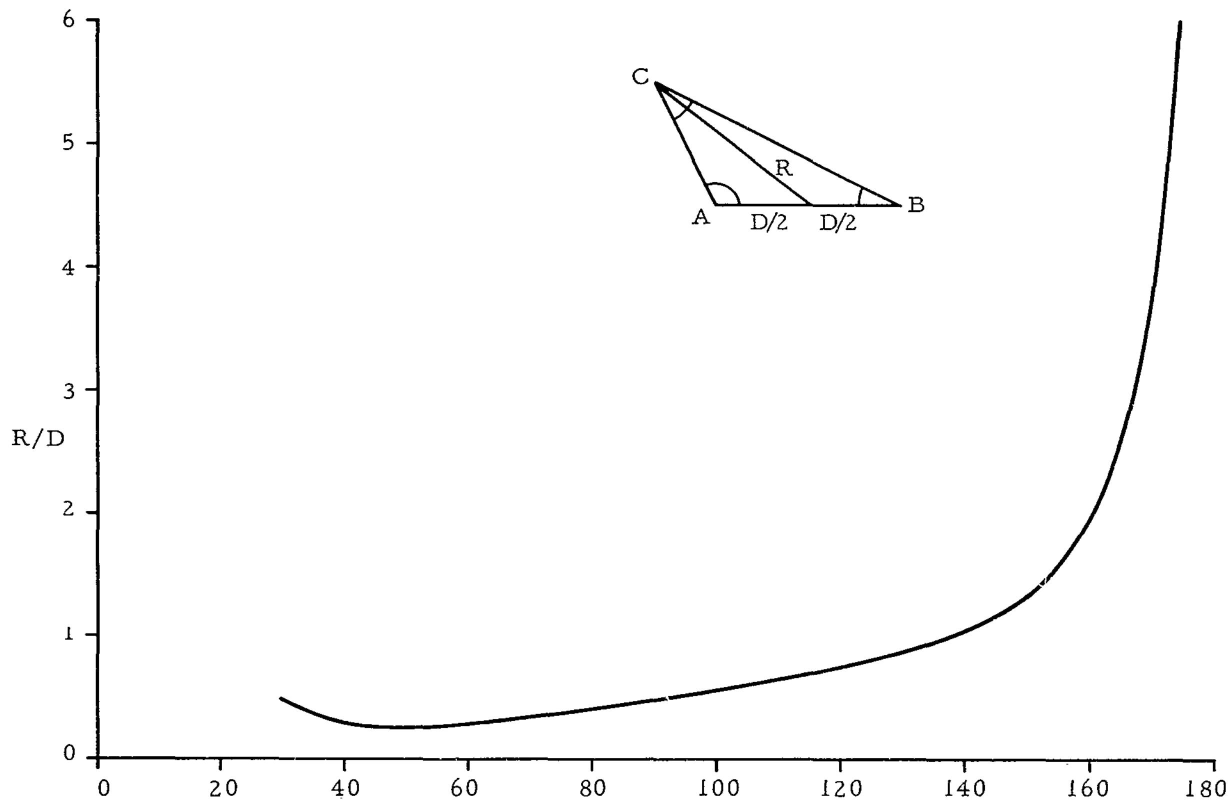

Figure 18

R/D vs (A + B) for B = 30°

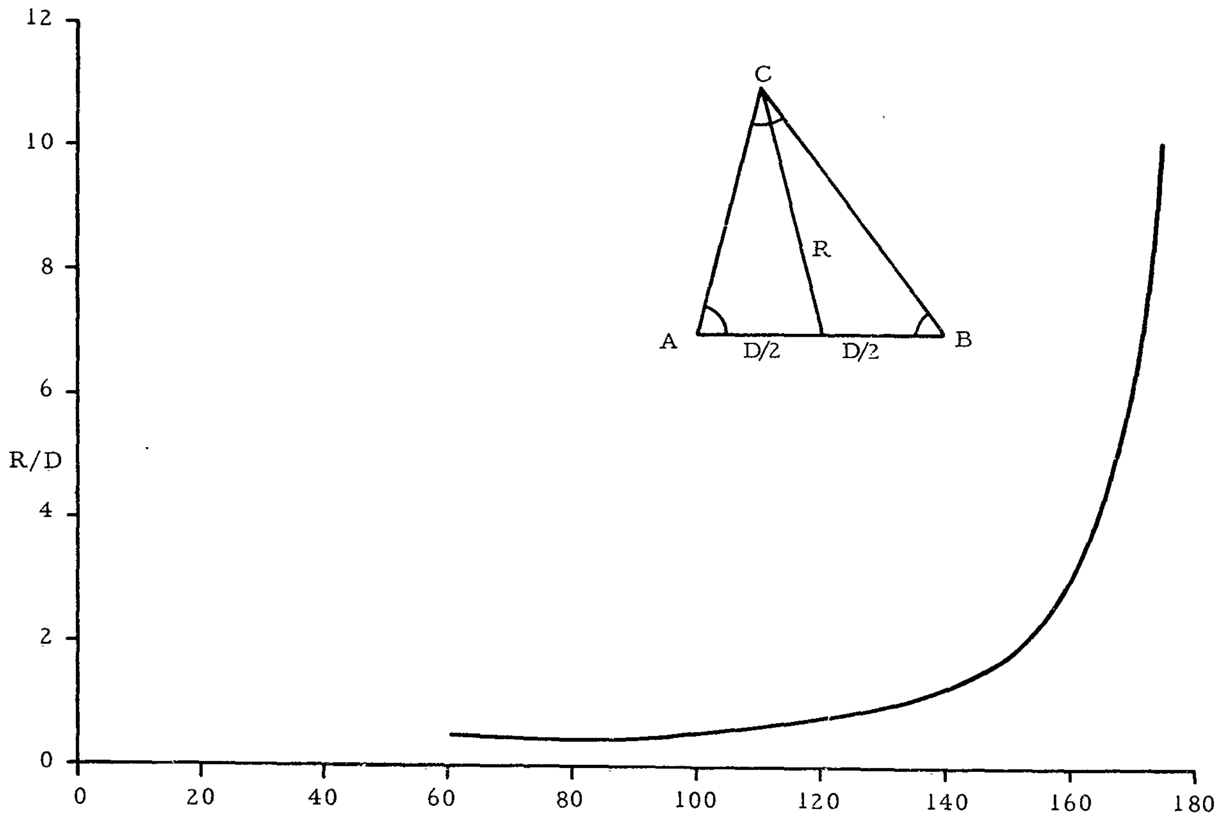

Figure 19

R/D vs (A + B) for B = 60°

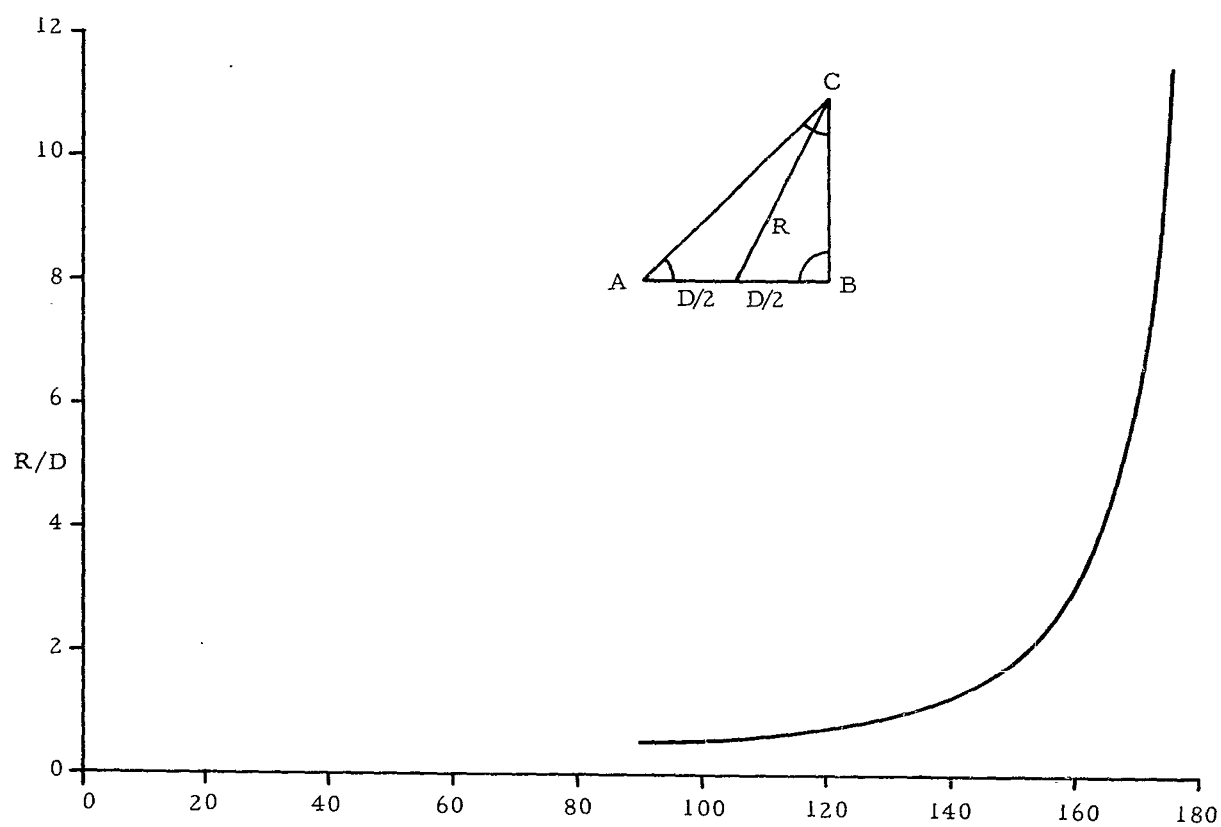

Figure 20
R/D vs (A + B) for B = 90°

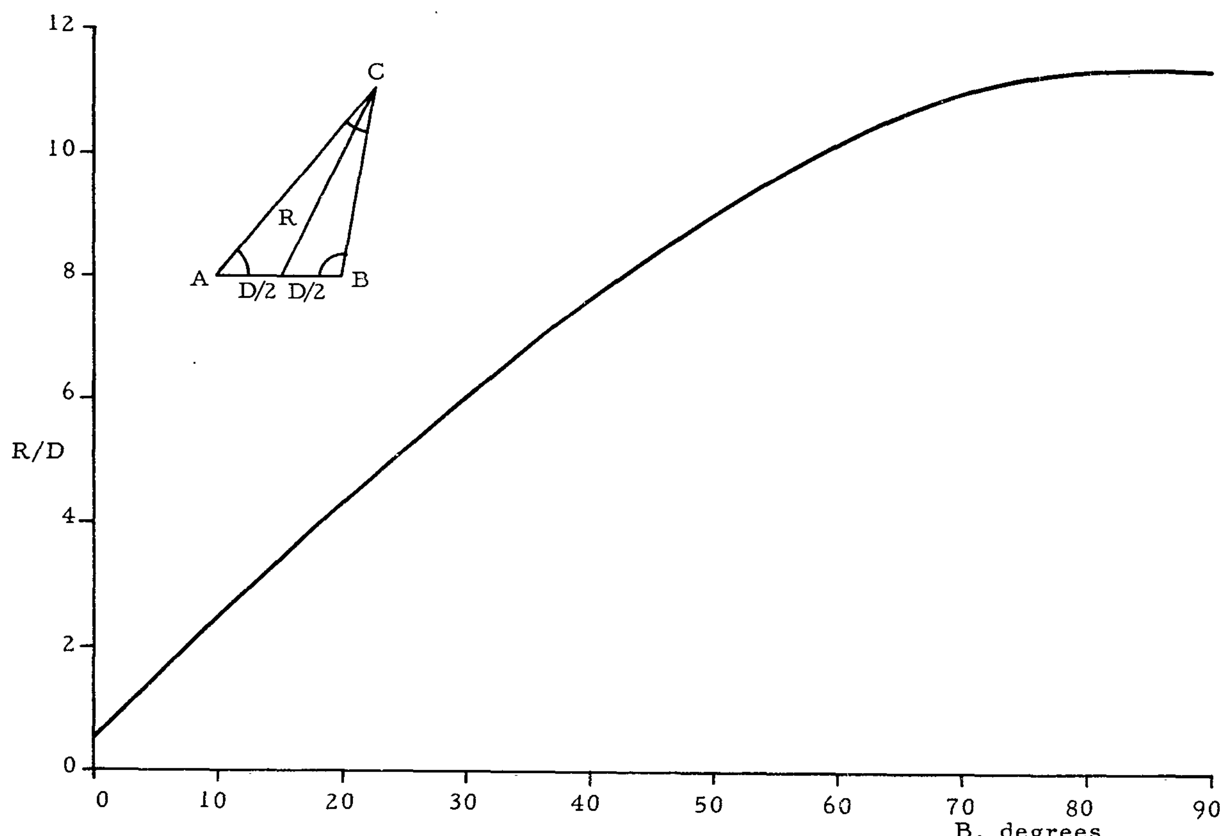

Figure 21

R/D vs B for Subtended Angle C = 5°

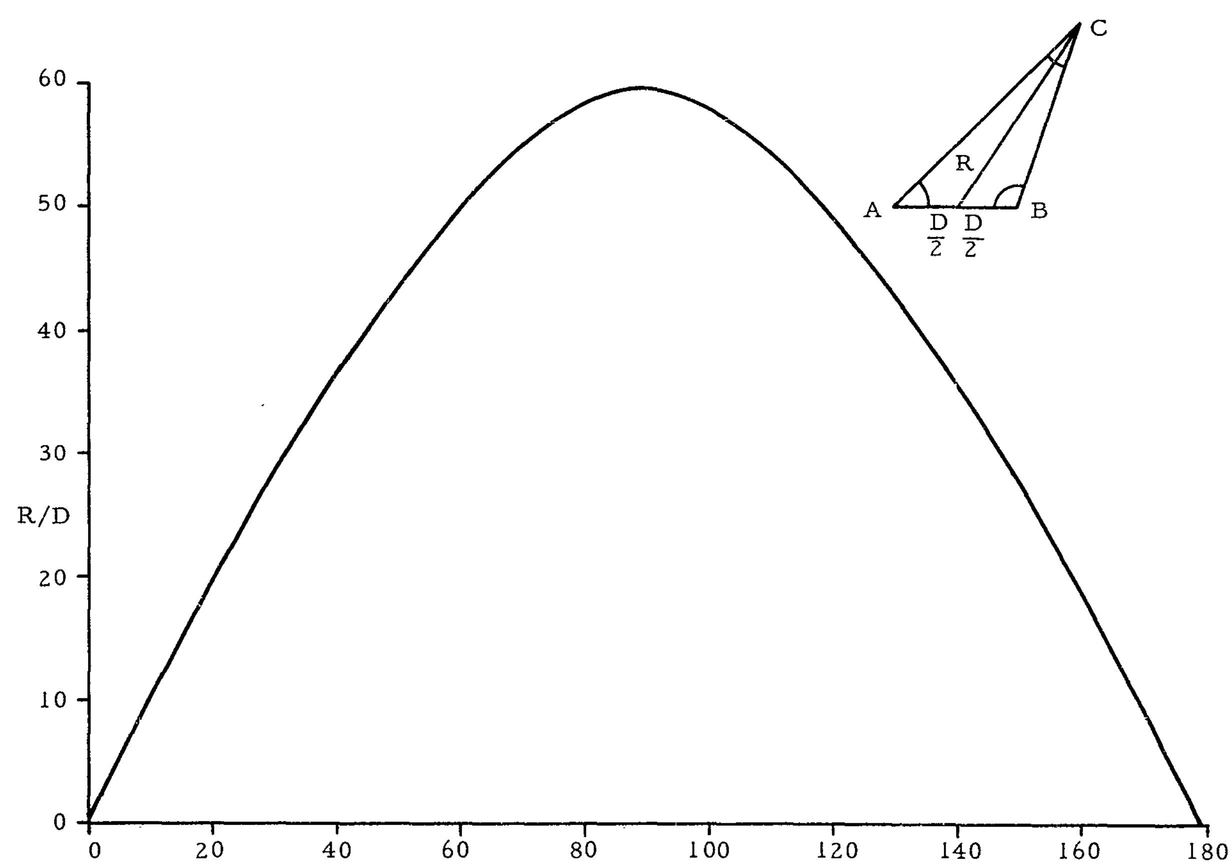

Figure 22

R/D vs B for Subtended Angle C = 1°

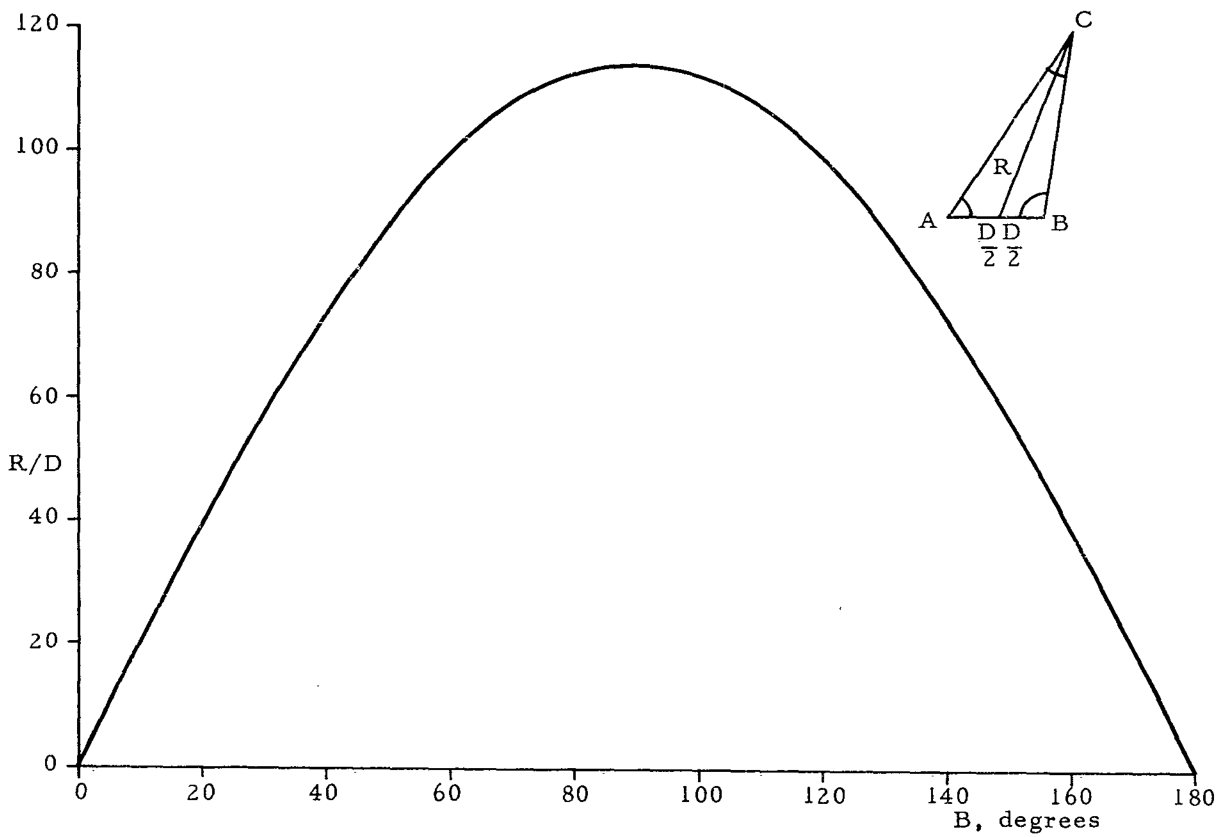

Figure 23

R/D vs B for Subtended Angle C = $1/2^{\circ}$

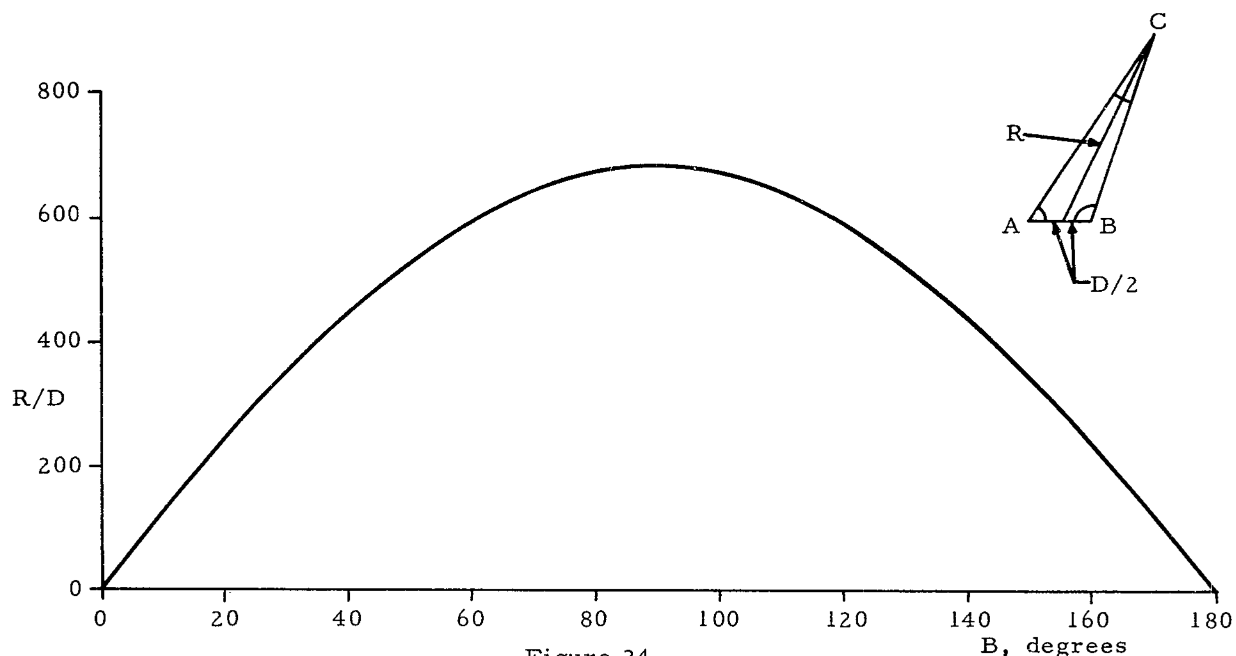

Figure 24

R/D vs B for Subtended Angle C = $0^{\circ}5'$

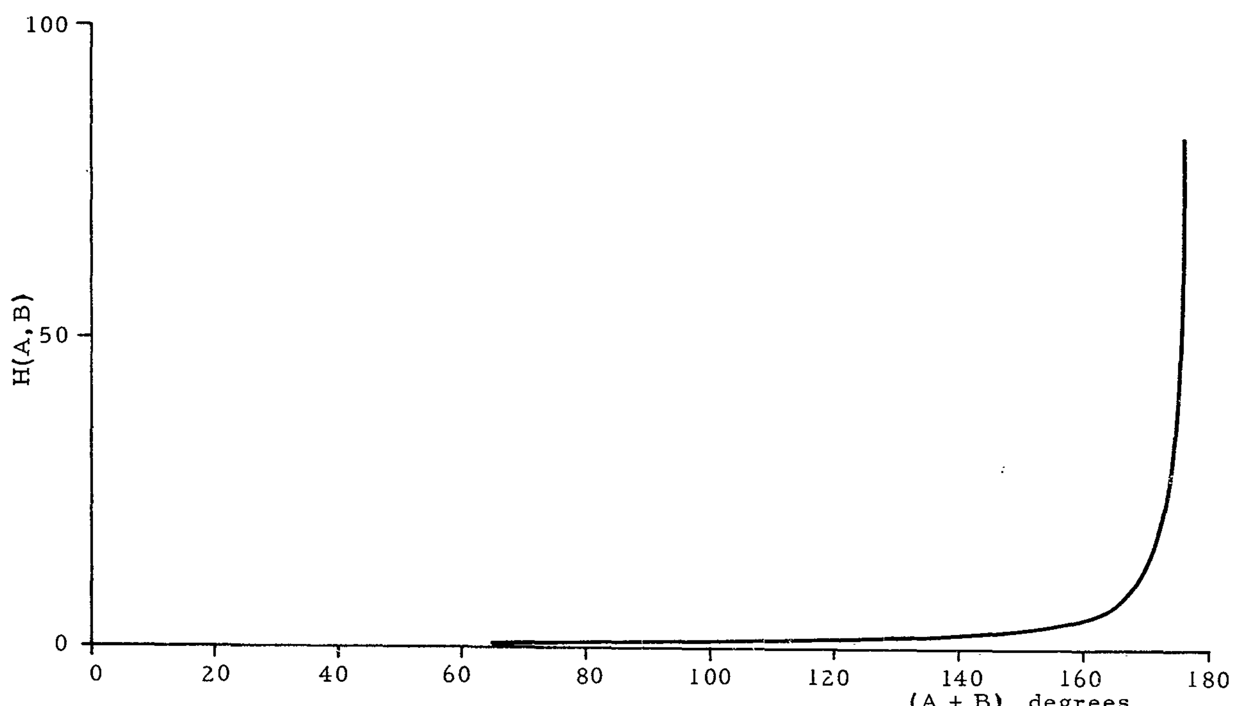

Figure 25

H(A, B) vs (A + B) for B = 5°

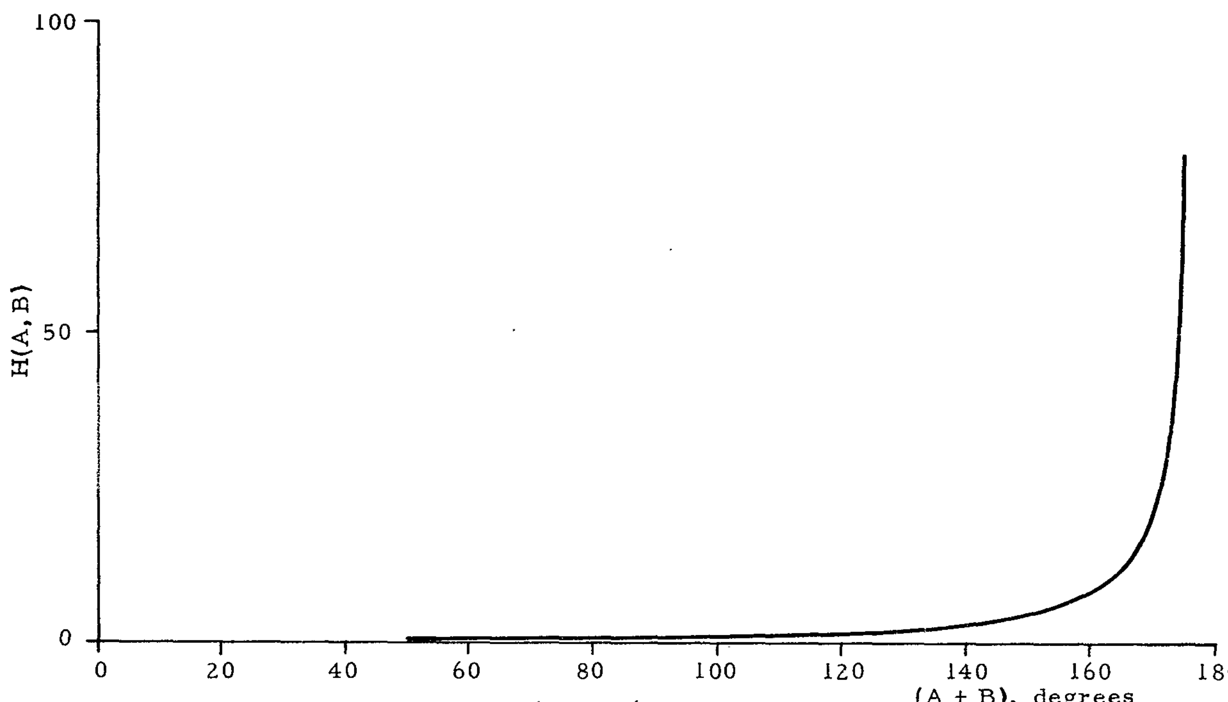

Figure 26

H(A,B) vs (A + B) for B = 15^{o}

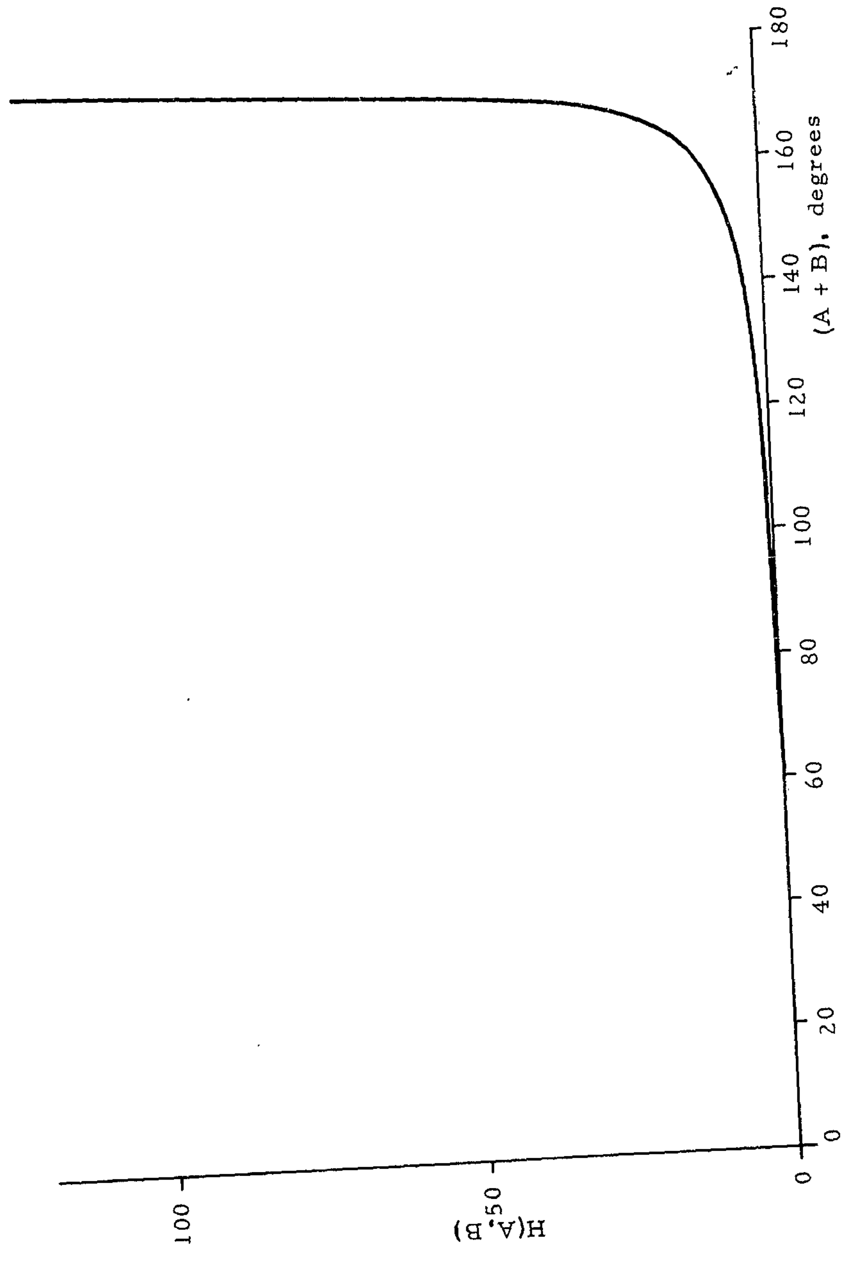

Figure 27

H(A,B) vs (A + B) for B = 30^o

$$\frac{dR}{RdA} \sim \frac{D}{R}\left[\frac{-2 \sin B \cos (A+B)}{\sin^2 (A+B)}\right] \sim 2/\sin (A+B)$$

If $C \leq 5^{\circ}$ and $dA << C/2$, this asymptotic relation gives good accuracy. One has also in this neighborhood

$$\left(\frac{dR}{RdA}\right)\left(\frac{D}{R}\right) \sim 2/\sin B$$

This last relation may be useful in determining the relations among the values for angular and ranging error, the range, and the length, D, of the base line. If all except one of these quantities is known, the remaining quantity may be found. Ratios could also be found for more than one unknown.

As an example of the use of the last equation, suppose it is given that $R = 2,000$ ft, $B = 15^{\circ}$, $dA = 3$ milliradians, $dR = 100$ ft. What length, D, of base line is consistent with this accuracy? From this equation $D = 927.3$ ft and this or larger values of D would give the required accuracy. A more accurate computation shows that this value of D corresponds to a range error of 132.7 ft; the last equation is thus a fairly good approximation even for angle C as large as 9°. It should always be verified that $2dA << 180^{\circ} - (A + B)$ before making use of this last relation. As another example, suppose $R = 2000$ ft, $D = 100$ ft, $dR = 100$ ft, $B = 15^{\circ}$. What size errors dA are permitted? Using the last equation, $dA = 0.32$ mils; angular errors of this size or smaller would give range errors within the 100-ft limit.

Problems of these two types may also be solved by use of the curves. Figures 28 through 30 show plots of R/D and dR/RdA vs (A+B) for B held constant. The value of (A+B) can be determined from one of these ratios, R/D or dR/RdA, and the other ratio may be read from the graph at the corresponding value of (A+B). The desired quantity can then be computed using the value of the ratio, together with the values of the remaining variables.

If the subtended angle C is not small, say $C > 5^{\circ}$, then the approximation $R/D \sim R_1/D$ is no longer valid. In this case more accurate expressions should be used for the differentials. On taking the differential of the R/D equation and multiplying by D/RdA, the result for dA = dB is

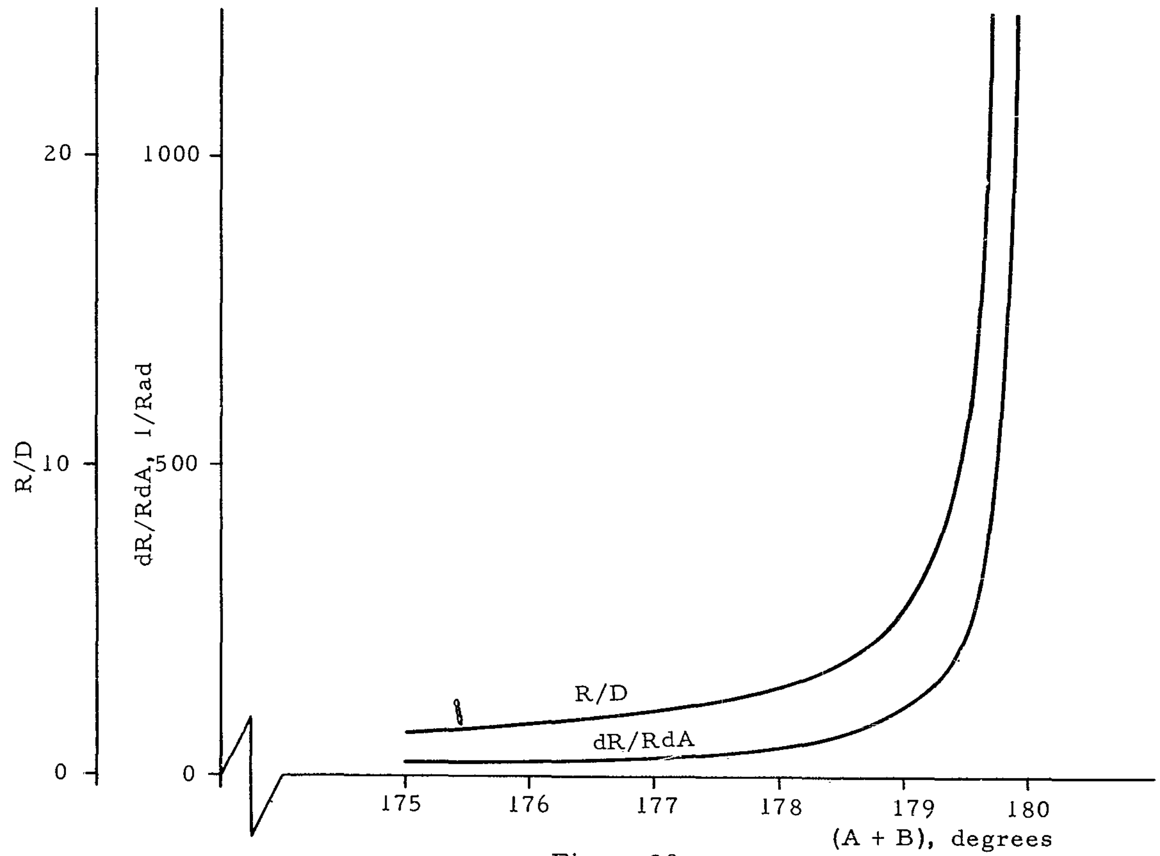

Figure 28

R/D and dR/RdA vs (A + B) for B = 5^{o}, dA = dB

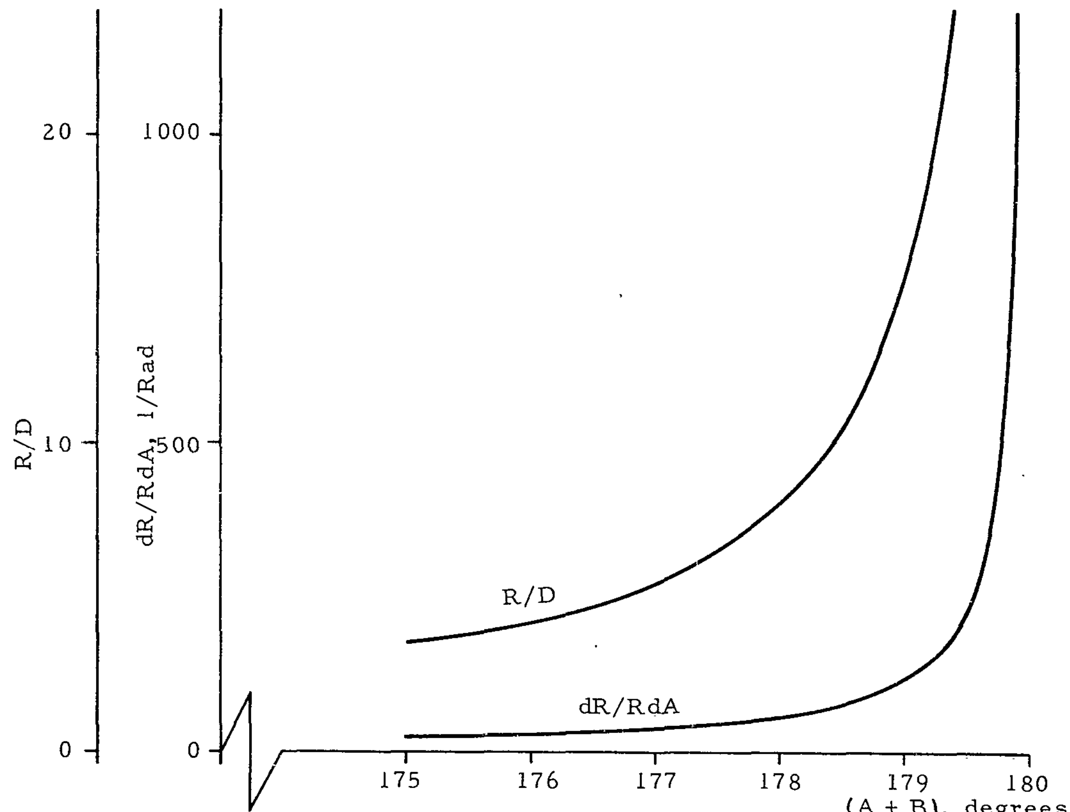

Figure 29

R/D and dR/RdA vs (A + B) for B = 15°, dA = dB

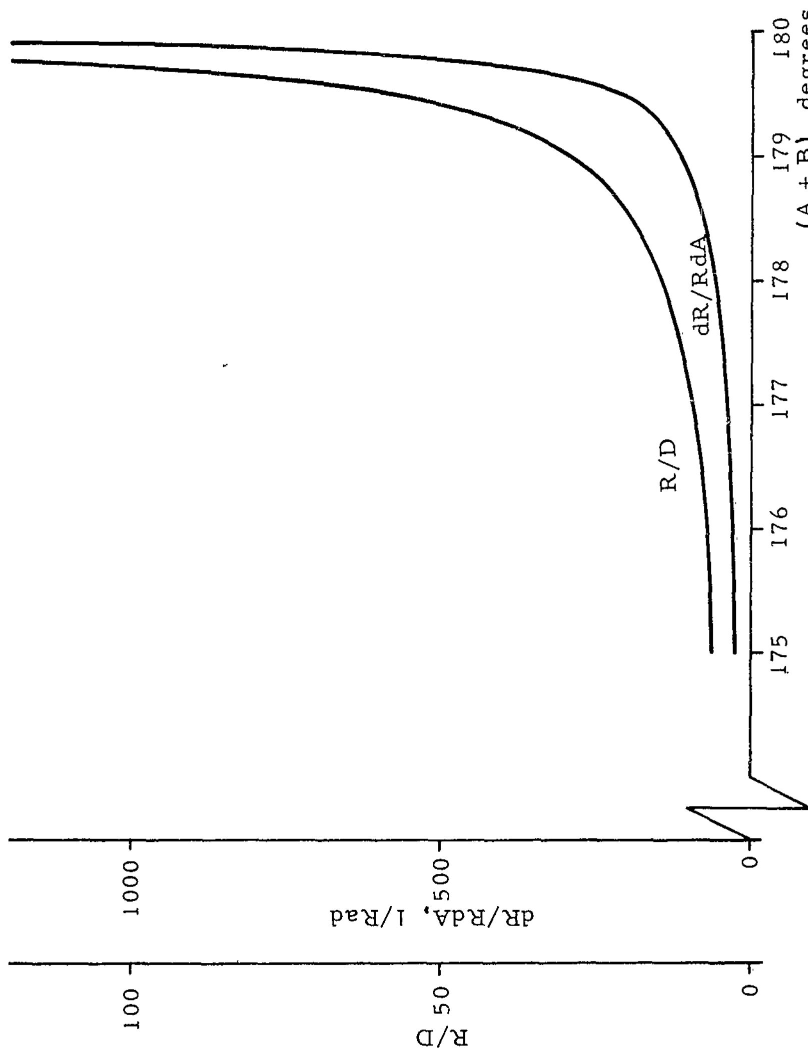

Figure 30

R/D and dR/RdA vs (A + B) for B = 30^{o}, dA = dB

$$\frac{dA}{RdA} = \frac{1}{2}\left(\frac{D}{R}\right)^2 \left\{ \left[\frac{\sin B}{\sin (A+B)} \left(\frac{-2 \sin B \cos (A+B)}{\sin^2 (A+B)} + \frac{\cos B}{\sin (A+B)} \right) \right. \right.$$

$$\left. \left. + \frac{\sin A}{\sin (A+B)} \left(\frac{-2 \sin A \cos (A+B)}{\sin^2 (A+B)} + \frac{\cos A}{\sin (A+B)} \right) \right] \right\}$$

This relation for $\frac{dR}{RdA}$ can be used with the equation for (R/D) to solve problems of the type of the two examples. One of the relations can be used to find (A + B), given B, and this value of (A + B) can then be used to find the unknown ratio which is to be solved for desired quantity (D, dA, or dR).

In conclusion, it may be stated that the most severe requirements are placed upon the accuracy of angles or the size of D when the subtended angle C is small or either A or B is small. Under such conditions, the use of this method of ranging and position finding by angle-only measurements is impractical. If range as well as angle information is available, the accuracy requirements would be greatly reduced; if redundant data are available, least squares or a similar method could be applied to improve the estimate of the position of the observed object.

Also, measurements of angles A and B alone could give good ranging and position data for short ranges, even with a short base line,, the subtended angle C being large. Such data could be extrapolated to give a trajectory. Unless the angular measurements are made with extreme accuracy, it appears probable that at extreme ranges of interest for scoring the extrapolated trajectory would be more reliable then the trajectory given by triangulation at these ranges.

Three-Station Measurements

The three-measuring-station geometry is as indicated in Figure 31.

The X_1, Y_1, X_2, Y_2, X_3, Y_3 axes lie in the plane determined by the three measuring stations. The distances ℓ_1 and ℓ_2 and the azimuth angle A_F are fixed previous to any measurements of point P, and the errors in the magnitudes of ℓ_1, ℓ_2, and A_F are assumed to be negligible in comparison to the errors introduced through the measurements of

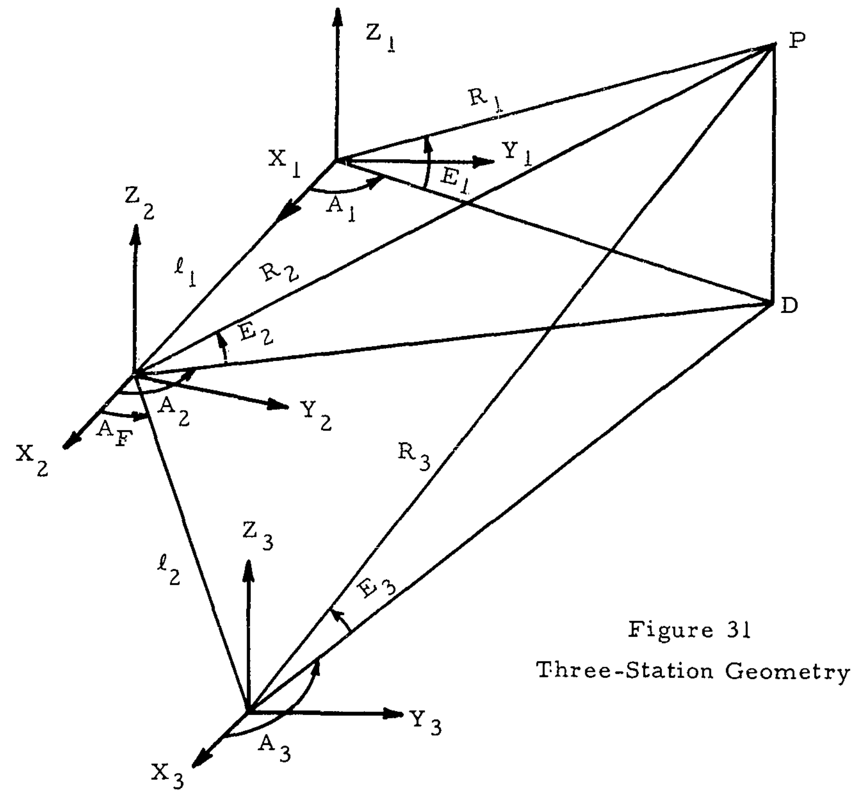

Figure 31

Three-Station Geometry

point P. Only a few special cases will be considered for the three-station situation, since any multiple-station measurements may be reduced ultimately to a two-station measurement.

The position of point P, for all cases, will be referred to a coordinate system located at the second measuring station, i.e., to the X_2, Y_2, Z_2 reference frame.

Case I. Measured Quantities: ℓ_1, ℓ_2, A_F, R_1, R_2, R_3

R_2 -- Measured Directly

$$A_2 = \tan^{-1} \left\{ \left[\frac{-M}{G} \right] \right\}$$

$$E_2 = \cos^{-1}\left\{\frac{K}{L}\left[1 + \frac{M^2}{G^2}\right]^{\frac{1}{2}}\right\}$$

where the sign of E_2 is indeterminate, and

$$G = \sin A_F$$

$$H = \cos A_F$$

$$J = \ell_1\left(R_3^2 - \ell_2^2 - R_2^2\right)$$

$$K = \ell_2\left(R_1^2 - \ell_1^2 - R_2^2\right)$$

$$L = 2R_2\,\ell_1\ell_2$$

$$M = H + \left(\frac{J}{K}\right)$$

Case II. Measured Quantities: ℓ_1, ℓ_2, A_F, A_1, A_2, E_3

$$R_2 = \frac{\ell_2}{J}\left\{H^2(1 + G^2) + G^2 J\,(J - 2HK)\right\}^{\frac{1}{2}}$$

A_2 -- Measured directly

$$E_2 = \tan^{-1}\left\{\frac{G}{H}\left[H^2 + J^2 - 2HJK\right]^{\frac{1}{2}}\right\}$$

where

$$G = \tan E_3$$

$$H = \ell_1 \sin A_1$$

$$J = \ell_2 \sin (A_2 - A_1)$$

$$K = \cos (A_2 - A_F)$$

Case III. Measured Quantities: ℓ_1, ℓ_2, A_F, A_1, A_3, E_2

$$R_2 = \frac{G}{K}\left\{H^2 + J^2 + 2HJL\right\}^{\frac{1}{2}}$$

$$A_2 = \tan^{-1}\left\{\frac{H[M+J]}{[HN + \ell_1 LJ]}\right\}$$

E_2 -- Measured directly

where

$$G = \ell_1 \sec E_2$$

$$H = \sin A_1$$

$$I = \cos A_1$$

$$J = \frac{\ell_2}{\ell_1} \sin (A_3 - A_F)$$

$$K = \sin (A_3 - A_1)$$

$$L = \cos (A_3 - A_1)$$

$$M = \sin A_3$$

$$N = \cos A_3$$

Case IV. Measured Quantities: ℓ_1, ℓ_2, A_F, A_2, A_3, E_1

$$R_2 = \frac{\ell_1}{J} \left\{ H^2(1 + G^2) + G^2 J (J + 2HK) \right\}^{\frac{1}{2}}$$

A_2 -- Measured directly

$$E_2 = \tan^{-1} \left\{ \frac{G}{H} [H^2 + J^2 + 2HJK]^{\frac{1}{2}} \right\}$$

where

$$G = \tan E_1$$

$$H = \ell_2 (A_3 - A_F)$$

$$J = \ell_1 \sin (A_3 - A_2)$$

$$K = \cos A_2$$

Weighting Redundant Data

The accuracy with which the position of a point in space may be determined with respect to some arbitrary reference frame is completely dependent upon the accuracy of the measuring devices being used. A quantity which is not measured directly, but which is

$$E_2 = \cos^{-1}\left\{\frac{K}{L}\left[1 + \frac{M^2}{G^2}\right]^{\frac{1}{2}}\right\}$$

where the sign of E_2 is indeterminate, and

$$G = \sin A_F$$

$$H = \cos A_F$$

$$J = \ell_1\left(R_3^2 - \ell_2^2 - R_2^2\right)$$

$$K = \ell_2\left(R_1^2 - \ell_1^2 - R_2^2\right)$$

$$L = 2R_2\,\ell_1\ell_2$$

$$M = H + \left(\frac{J}{K}\right)$$

Case II. Measured Quantities: ℓ_1, ℓ_2, A_F, A_1, A_2, E_3

$$R_2 = \frac{\ell_2}{J}\left\{H^2(1 + G^2) + G^2 J\,(J - 2HK)\right\}^{\frac{1}{2}}$$

A_2 -- Measured directly

$$E_2 = \tan^{-1}\left\{\frac{G}{H}\left[H^2 + J^2 - 2HJK\right]^{\frac{1}{2}}\right\}$$

where

$$G = \tan E_3$$

$$H = \ell_1 \sin A_1$$

$$J = \ell_2 \sin(A_2 - A_1)$$

$$K = \cos(A_2 - A_F)$$

Case III. Measured Quantities: ℓ_1, ℓ_2, A_F, A_1, A_3, E_2

$$R_2 = \frac{G}{K}\left\{H^2 + J^2 + 2HJL\right\}^{\frac{1}{2}}$$

$$A_2 = \tan^{-1}\left\{\frac{H[M+J]}{[HN + \ell_1 IJ]}\right\}$$

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computed from several measured quantities, will also be subject to the errors introduced through the measuring devices. Assume that X is a quantity which is to be computed from several measured quantities, $Z_1, Z_2, \ldots, Z_n$, and that a functional relationship between X and the measured Z's exists. This relationship could be written as

$$X = X(Z_1, Z_2, \ldots, Z_n)$$

If the errors in the measured quantities are sufficiently small, the error in the computed quantity, X, is given approximately by the linear relation

$$dX = \frac{\partial X}{\partial Z_1} dZ_1 + \frac{\partial X}{\partial Z_2} dZ_2 + \ldots + \frac{\partial X}{\partial Z_n} dZ_n$$

where dZ_i is the error in the measured quantity Z_i. Further, if the error distribution in the measured quantities is such that the greatest majority of the errors are sufficiently small for the Taylor expansion to be approximately valid, and if the errors are independent, then

$$\frac{1}{N} \sum (dX)^2 = \frac{1}{N} \sum \left(\frac{\partial X}{\partial Z_1}\right)^2 \left(dZ_1\right)^2 + \left(\frac{\partial X}{\partial Z_2}\right)^2 \left(dZ_2\right)^2 + \ldots + \left(\frac{\partial X}{\partial Z_n}\right)^2 \left(dZ_n\right)^2$$

or the variance, σ_X^2, is

$$\sigma_X^2 = \left(\frac{\partial X}{\partial Z_1}\right)^2 \sigma_{Z_1}^2 + \left(\frac{\partial X}{\partial Z_2}\right)^2 \sigma_{Z_2}^2 + \ldots + \left(\frac{\partial X}{\partial Z_n}\right)^2 \sigma_{Z_n}^2$$

If the dZ's have a mean of zero, the computed quantity dX will have a mean of zero also. No information is required about the distribution of errors to get a mean of zero; however, for obtaining the approximate average square of error, additional information about the distribution is required.

It is to be stressed that once the functional relationships between the measured quantities and the quantities to be computed have been established, the differentials of these relationships may be used to establish limiting conditions on the variances that can be tolerated. If, for example, in the above relationship, $X = X(Z_1, Z_2, \ldots, Z_n)$, the average square values of the errors involved in measuring the Z_i are known, then the approximate variance or average square value of error in the computed quantity X is obtained from

$$\sigma_X^2 = \left(\frac{\partial X}{\partial Z_1}\right)^2 \sigma_1^2 + \left(\frac{\partial X}{\partial Z_2}\right) \sigma_2^2 + \ldots + \left(\frac{\partial X}{\partial Z_n}\right)^2 \sigma_n^2$$

The measurements of the Z_i's are assumed to be uncorrelated, so that for N measurements

$$\left(\overline{dZ_i dZ_j}\right) = \frac{1}{N} \sum_{k=1}^{N} (dZ_i dZ_j)_k \cong 0$$

The coefficients in the σ_X^2 equation will in general turn out to be n rather different functions of the dependent variables. If the variances σ_i^2 are equal, only the maximum sum of the coefficients will be needed for error-study purposes. In general, however, it will be necessary to compute several "extreme" sets of coefficients. If the coefficients are bounded, they can be used to set limits on the variances such that a specified σ_X^2 will not be exceeded; unless an arbitrary distribution of the allowable error is made, a trial-and-error process, with some attention given to the practical limits of the equipment, will be needed in general. Of course if the σ_i^2's are known to have particular values, then the maximum (or minimum) variance of the computed quantity, X, can be established.

Some comments on the theorem by Tschebyscheff might be of interest. This theorem states that "the probability of obtaining a value of the standardized variable which is numerically less than or equal to a specified number k is larger than $1 - \frac{1}{k^2}$." That is,

$$P\left\{|X - \overline{X}| \leq k\sigma\right\} > 1 - \frac{1}{k^2}$$

Some numerical values of this function are as follows:

k	1.0	1.41	1.58	1.83	2.0	2.24	3.0	3.16	10
$1 - \frac{1}{k^2}$	0	.50	.60	.70	.75	.80	.889	.90	.99

From this table, according to the theorem, it can be seen that independent of the distribution of errors and independent of the magnitude of σ, the probability of measuring X to within say 1.83 σ units of the correct value $\overline{X}$ is greater than 70 percent. Thus some

limits on the reliability of the measured quantities may be established.

The probability of an error not being greater than a given size may be stated in terms of the variance, according to Tschebyscheff's theorem. If the distribution of the errors is known, the bounds of these probability relations may be made more precise. For the normal distribution the probable error is 0.6745σ. This statement is to be interpreted as giving the size of the error such that one half the errors are less than that size and the other half are greater. According to common usage, experimental results may be given as the measured values plus or minus the probable errors.

If a requirement were made that the size of the errors be less than B with a probability C, Tschebyscheff's theorem may be applied to show that the requirement is satisfied for any distribution of errors with a standard deviation, σ, such that

$$k\sigma \leq B$$

$$C = P\{|X - \overline{X}| \leq k\sigma\} > 1 - \frac{1}{k^2}$$

The last relation may be solved to give $k < 1/\sqrt{1 - C}$, so that both equations are satisfied if we have $\sigma \leq B\sqrt{1 - C}$. More precise bounds on σ for such a requirement can be obtained if the distribution of errors is known. For example, with a normal distribution, the following table applies for this requirement:

C	0.5	0.6	0.7	0.8	0.9
$\sigma \leq$	1.481B	1.187B	0.965B	0.780B	0.608B

Variance relations for the errors in a computed quantity in terms of the variances of the observed quantities have been discussed. When the variance of the computed quantity is prescribed, limits are placed on the variances of the observed quantities. The allowable range of instrument errors could be apportioned among the different observed quantities in many ways. An arbitrary rule that has been proposed to resolve this indeterminate problem is to apportion the errors such that the permitted errors in the observed quantities have equal effects on the errors of the computed quantities. Thus, if X is the computed quantity, $\alpha_1, \alpha_2, \ldots, \alpha_N$ are observed quantities, and the variance equation is

$$\sigma_X^2 = A_1\sigma_{\alpha_1}^2 + A_2\sigma_{\alpha_2}^2 + \ldots + A_N\sigma_{\alpha_N}^2$$

choose $A_1\sigma_{a_1}^2 = A_2\sigma_{a_2}^2 = \ldots = A_N\, \sigma_{a_N}^2 \leq \frac{\sigma_X^2}{N}$. As the quantities A_i are not constant but are functions of the geometry, a conservative rule would put $\sigma_{a_i}^2 \leq \sigma_X^2/NA_{i_{Max}}$. The extension to the case of several dependent variables is obvious.

We will now consider weighting factors for computed quantities when redundant data have been obtained. We shall consider the case in which the quantity X may be uniquely specified by any three of four measurable quantities. This means that four functional relationships exist between the quantity X and the measured quantities, Z_1, Z_2, Z_3, and Z_4. We shall indicate these relationships as

$$X_1 = X_1\,(Z_2,\ Z_3,\ Z_4)$$

$$X_2 = X_2\,(Z_1,\ Z_3,\ Z_4)$$

$$X_3 = X_3\,(Z_1,\ Z_2,\ Z_4)$$

$$X_4 = X_4(Z_1,\ Z_2,\ Z_3)$$

If small errors are assumed, the differential errors in X_i are given by

$$dX_i = \sum_{j \neq i}^{4} \frac{\partial X_i}{\partial Z_j}\, dZ_j .$$

and if the two measured quantities Z_ℓ and Z_m are uncorrelated (see Supplement), the expression

$$\overline{dX_i\, dX_j} = \sum_{m \neq i}^{4} \sum_{\ell \neq j}^{4} \frac{\partial X_i}{\partial Z_m} \frac{\partial X_j}{\partial Z_\ell}\ \overline{dZ_m\, dZ_\ell}$$

may be reduced to

$$\overline{dX_i\, dX_j} = \sum_{\ell \neq i \neq j}^{4} \frac{\partial X_i}{\partial Z_\ell} \frac{\partial X_j}{\partial Z_\ell}\ \overline{(dZ_\ell)^2}$$

The computed quantities $X_i (i = 1, 2, 3, 4)$ are now to be weighted to get the best estimate of X. Thus, if

$$\overline{X} = \sum_{i=1}^{4} W_i X_i$$

where W_i is the value of the weight for the computed X_i, then the quantity

$$\overline{(dX)^2} = \sum_{i=1}^{4} \sum_{j=1}^{4} W_i W_j \overline{dX_i \, dX_j}$$

must be minimized with respect to the weighting factors, subject to the constraint

$$\sum_{i=1}^{4} W_i = 1$$

If the undetermined multiplier, λ, is introduced, this constraint may be written as

$$2\lambda \left(1 - \sum_{i=1}^{4} W_i\right) = 0$$

so that

$$\overline{(dX)^2} = \sum_{i=1}^{4} \sum_{j=1}^{4} W_i W_j \overline{dX_i \, dX_j} - 2\lambda \left(1 - \sum_{i=1}^{4} W_i\right)$$

To minimize $\overline{(dX)^2}$ with respect to the weighting factors, W_i, we equate $\frac{\partial \overline{(dX)^2}}{\partial W_i}$, (i = 1, 2 , 3, 4), to zero. Thus

$$\frac{\partial \overline{(dX)^2}}{\partial W_i} = \sum_{j=1}^{4} W_j \overline{dX_i\, dX_j} + \lambda = 0, \quad (i = 1, 2, 3, 4)$$

These four equations in the weighting factors have a solution

$$W_i = \lambda B_i(dX_1, dX_2, dX_3, dX_4), \quad (i = 1, 2, 3, 4)$$

where the B_i's are obtained from diagonalizing the $\overline{dX_i\, dX_j}$ matrix. Returning to the normalization constraint

$$\sum_{i=1}^{4} W_i = \lambda(B_1 + B_2 + B_3 + B_4) = 1$$

the value of λ is found to be

$$\lambda = \frac{1}{\sum_{i=1}^{4} B_i}$$

and finally the weighting factors are given by

$$W_i = \frac{B_i}{\sum_{j=1}^{4} B_j}, \quad (i = 1, 2, 3, 4)$$

Applications and Conclusions

As an application of the equations derived in the geometry portion of this report and as an application of the method for weighting the results, let us consider the following example. Assume a two-station situation in which the base line ℓ is known exactly; the azimuth, elevation, and range of the point P are measured from one station (call these A_1, E_1, and ρ_1) while only the azimuth and elevation of the point are measured from the second station (call these A_2 and E_2). It is then possible to apply the two-station formulas given in Cases I

and III above. Only the expressions relating range to the measured quantities will be used in this example. We have from Case I,

$$R_1 = \ell \left\{ \frac{\cos^2 E_1 \sin^2(A_2 - A_1) + 4\sin^2 A_2 - \cos A_1 \sin A_2 \cos^2 E_1 \sin(A_2 - A_1)}{4\cos^2 E_1 \sin^2(A_2 - A_1)} \right\}^{\frac{1}{2}}$$

and from Case III

$$R_2 = \left\{ \rho_1^2 + \ell^2\left(-\frac{3}{4} + \cos^2 A_2 \cos^2 E_2\right) + \ell \cos A_2 \cos E_2 \sqrt{\rho_1^2 - \ell^2(1 - \cos^2 A_2 \cos^2 E_2)} \right\}^{\frac{1}{2}}.$$

From these two expressions it is desirable to find weighting factors which give a best estimate of R, assuming that the errors (or more exactly, the variances) for the measured quantities are known. That is, we wish to find

$$\overline{R} = W_1 R_1 + W_2 R_2$$

such that $\overline{(dR)^2}$ is a minimum, and subject to the constraint

$$W_1 + W_2 = 1$$

In a straightforward manner, the differential errors in R_1 and R_2 are found to be

$$dR_1 = K_1\, dA_1 + K_2\, dA_2 + K_3\, dE_1$$

and

$$dR_2 = L_1\, d\rho_1 + L_2\, dA_2 + L_3\, dE_2$$

where

$$K_1 = \left(\frac{\partial R_1}{\partial A_1}\right) = M \left[\frac{\sin A_1 \sin A_2}{4\sin(A_2 - A_1)} + \frac{2\sin^2 A_2 \cos(A_2 - A_1)}{\cos^2 E_1 \sin^3(A_2 - A_1)} - \frac{\sin A_2 \cos A_1 \cos(A_2 - A_1)}{4\sin^2(A_2 - A_1)} \right]$$

$$K_2 = \left(\frac{\partial R_1}{\partial A_2}\right) = M\left[\frac{2\sin A_2 \cos A_2}{\cos^2 E_1 \sin^2(A_2 - A_1)} - \frac{\cos A_2 \cos A_1}{\sin(A_2 - A_1)4} - \frac{2\sin^2 A_2 \cos(A_2 - A_1)}{\cos^2 E_1 \sin^3(A_2 - A_1)} + \frac{\sin A_2 \cos A_1 \cos(A_2 - A_1)}{4\sin^2(A_2 - A_1)}\right]$$

$$K_3 = \left(\frac{\partial R_1}{\partial E_1}\right) = M\left[\frac{2\sin^2 A_2 \sin E_1}{\sin^2(A_2 - A_1)\cos^3 E_1}\right]$$

With

$$M = \frac{\ell}{2}\left\{\frac{1}{4} + \frac{\sin^2 A_2}{\cos^2 E_1 \sin^2(A_2 - A_1)} - \frac{\sin A_2 \cos A_1}{4\sin(A_2 - A_1)}\right\}^{-\frac{1}{2}}$$

and

$$L_1 = \left(\frac{\partial R_2}{\partial \rho_1}\right) = N\left[2\rho_1 + \frac{\ell\rho_1 \cos A_2 \cos E_2}{\sqrt{\rho_1^2 - \ell^2(1 - \cos^2 A_2 \cos^2 E_2)}}\right]$$

$$L_2 = \left(\frac{\partial R_2}{\partial A_2}\right) = N\left[-\ell \sin A_2 \cos E_2 \sqrt{\rho_1^2 - \ell^2(1 - \cos^2 A_2 \cos^2 E_2)}\left\{1 + \frac{\ell \cos A_2 \cos E_2}{\sqrt{\rho_1^2 - \ell^2(1 - \cos^2 A_2 \cos^2 E_2}}\right\}^2\right]$$

$$L_3 = \left(\frac{\partial R_2}{\partial E_2}\right) = N \left[-\ell \sin E_2 \cos A_2 \sqrt{\rho_1^2 - \ell^2 \left(1 - \cos^2 A_2 \cos^2 E_2\right)} \left\{ 1 + \frac{\ell \cos A_2 \cos E_2}{\sqrt{\rho_1^2 - \ell^2 \left(1 - \cos^2 A_2 \cos^2 E_2\right)}} \right\}^2 \right]$$

with

$$N = \frac{1}{2} \left\{ \rho_1^2 + \ell^2 \left(- \frac{3}{4} + \cos^2 A_2 \cos^2 E_2 \right) + \ell \cos A_2 \cos E_2 \sqrt{\rho_1^2 - \ell^2 \left(1 - \cos^2 A_2 \cos^2 E_2\right)} \right\}^{-\frac{1}{2}}$$

From the previous development, we take the derivative of

$$\overline{(dR)^2} = \sum_i \sum_j W_i W_j \overline{dR_i \, dR_j} - 2\lambda \left(1 - \sum_k W_k\right)$$

$$= W_1^2 \overline{(dR_1)^2} + 2W_1 W_2 \overline{dR_1 \, dR_2} + W_2^2 \overline{(dR_2)^2} - 2\lambda(1 - W_1 - W_2)$$

$$= W_1^2 \left(K_1^2 \sigma_{A_1}^2 + K_2^2 \sigma_{A_2}^2 + K_3^2 \sigma_{E_1}^2\right) + 2W_1 W_2 \left(K_2 L_2 \sigma_{A_2}^2\right)$$

$$+ W_2^2 \left(L_1^2 \sigma_{\rho_1}^2 + L_2^2 \sigma_{A_2}^2 + L_3^2 \sigma_{E_2}^2\right) - 2\lambda \left(1 - W_1 - W_2\right)$$

with respect to the weighting factors, to obtain the two equations

$$\left(K_1^2 \sigma_{A_1}^2 + K_2^2 \sigma_{A_2}^2 + K_3^2 \sigma_{E_1}^2\right) W_1 + \left(K_2 L_2 \sigma_{A_2}^2\right) W_2 = -\lambda$$

$$\left(K_2 L_2 \sigma_{A_2}^2\right) W_1 + \left(L_1^2 \sigma_{\rho_1}^2 + L_2^2 \sigma_{A_2}^2 + L_3^2 \sigma_{E_2}^2\right) W_2 = -\lambda$$

Through use of the normalization condition, these equations may be solved to give

$$W_1 = \frac{1}{D}\left[L_1^2\sigma_{\rho_1}^2 + \left(L_2^2 - K_2L_2\right)\sigma_{A_2}^2 + L_3^2\sigma_{E_2}^2\right]$$

$$W_2 = \frac{1}{D}\left[K_1^2\sigma_{A_1}^2 + \left(K_2^2 - K_2L_2\right)\sigma_{A_2}^2 + K_3^2\sigma_{E_1}^2\right]$$

where

$$D = K_1^2\sigma_{A_1}^2 + \left(K_2 - L_2\right)^2\sigma_{A_2}^2 + K_3^2\sigma_{E_1}^2 + L_1^2\sigma_{\rho_1}^2 + L_3^2\sigma_{E_2}^2$$

To carry the example further the calculation of numerical values for the variances (or average squares of errors) of R_1, R_2, and the weighted value of R will be outlined. The values of the parameters chosen have no special significance. Values for ℓ, ρ_1, E_1, and A_1 were assumed, and values of E_2 and A_2 were then computed to give "near" closure of the various triangles. These values are

ℓ = 100 feet

ρ_1 = 2000 feet

E_1 = 20°

A_1 = 60°

E_2 = $20^\circ 29'$

A_2 = $62^\circ 42'$

For this particular choice of parameters, it is found that

R_1 = 1984.69 feet

R_2 = 1980.79 feet

These values are in very good agreement. If the values of E_2 and A_2 had not been chosen to give "near" closure of the triangles, a much larger disagreement between R_1 and R_2 would have resulted.

The other computed quantities of interest from this set of parameters are

$$K_1 = 4.2610 \times 10^4 \text{ feet}$$

$$K_2 = -4.1438 \times 10^4 \text{ feet}$$

$$K_3 = 7.3916 \times 10^2 \text{ feet}$$

$$L_1 = 9.9884 \times 10^{-1}$$

$$L_2 = 4.0197 \times 10^1 \text{ feet}$$

$$L_3 = 1.5828 \times 10^1 \text{ feet}$$

The variances for the distance and angular measurements are then assumed to be

$$\sigma^2(\rho_1) = 10^2 \text{ft}^2$$

$$\sigma^2 \text{ (all angles)} = 9 \times 10^{-6}$$

This gives the variances for R_1 and R_2 as

$$\sigma^2_{R_1} = 31799. \text{ ft}^2$$

$$\sigma^2_{R_2} = 99.786 \text{ ft}^2$$

By the process outlined above, the weighting parameters are found to have the values

$$W_1 = .00359$$

$$W_2 = .99641$$

The weighted value of R,

$$\overline{R} = W_1 R_1 + W_2 R_2$$

is then found to be

$$\overline{R} = 1980.8 \text{ ft}$$

with a variance of

$$\sigma_{\overline{R}}^{2} = 99.3733 \text{ ft}^2$$

By direct comparison then, it is seen that

$$\sigma_{\overline{R}}^{2} < \sigma_{R_2}^{2} < \sigma_{R_1}^{2}$$

which points out the value of the weighted average for the computed quantity, in this case R.

For emphasis, it is pointed out again that when a particular measuring scheme has been set up and the variances have been established for the measured quantities, then the variances of the computed quantities may be calculated. Or, if the maximum allowable variance is specified for a particular computed quantity, limits may be established for the measured quantities; i.e., the maximum errors that the measuring devices could have for the error of the computed quantity to be less than some given amount could be established. To determine these limits for the errors of the measuring devices, however, the functional relations given previously between the computed and measured quantities and the differentials of these functional relations are required. The method of weighting parameters is required only when a redundancy of data occurs.

SUPPLEMENT

PROPAGATION OF COVARIANCE OF ERRORS

Suppose certain variables y_k, $k = 1, 2, \ldots, m$, functions of another set of variables x_s, $s = 1, 2, \ldots, r$, are given as functional relations $y_k = y_k(x_1, x_2, \ldots, x_r)$. Linear relations for small errors in y due to small errors in x are then given by the differentials,

$$dy_k = \sum_j \frac{\partial y_k}{\partial x_j} dx_j \tag{1}$$

These errors may be viewed in a statistical sense if the linear error relations (1) hold for the great bulk of the errors, so that these relations may be averaged over the errors. If all the systematic errors and biases have been removed so that the errors dy_k and dx_j appearing in (1) are accidental errors, it may be inferred that the average errors are zero for both x and y.

The addition of a second subscript t to the quantities occurring in equation (1) may be made to indicate the results of the t-th experiment in measuring the errors.

$$dy_{kt} = \sum_i \frac{\partial y_k}{\partial x_i} dx_{it} \tag{2}$$

$$dy_{st} = \sum_j \frac{\partial y_s}{\partial x_j} dx_{jt} \tag{3}$$

The covariance of the errors in y may be obtained by taking the product of equations (2) and (3), and averaging over all the experiments.

$$dy_{kt}\, dy_{st} = \sum_{i,j} \frac{\partial y_k}{\partial x_i} \frac{\partial y_s}{\partial x_j} dx_{it}\, dx_{jt} \tag{4}$$

$$\lim_{h \to \infty} \frac{1}{h} \sum_{t=1}^{h} dy_{kt}\, dy_{st} = \lim_{h \to \infty} \frac{1}{h} \sum_{t=1}^{h} \sum_{i,j} \frac{\partial y_k}{\partial x_i} \frac{\partial y_s}{\partial x_j} dx_{it}\, dx_{jt} \tag{5}$$

or

$$\sigma_{y_k y_s} = \sum_{i,j} \frac{\partial y_k}{\partial x_i} \frac{\partial y_s}{\partial x_j} \sigma_{x_i x_j} \tag{6}$$

Equation (6) is a relation of the propagation of covariance of errors. If the errors dx_i and dx_j are independent for $i \neq j$, then $\sigma_{x_i x_j}$ reduces to $\sigma^2_{x_i} \delta^j_i$, where δ^j_i is the Kronecker delta, and equation (6) reduces to

$$\sigma_{y_k y_s} = \sum_{i} \left(\frac{\partial y_k}{\partial x_i}\right) \left(\frac{\partial y_s}{\partial x_i}\right) \sigma^2_{x_i} \tag{7}$$

This gives the covariance of errors in y in the event the errors in x are independent. In particular, for $y_k = y_s$, there results for this case of independent errors in x,

$$\sigma^2_{y_k} = \sum_{i} \left(\frac{\partial y_k}{\partial x_i}\right)^2 \sigma^2_{x_i} \tag{8}$$

Equations (6), (7), and (8) may be useful for interpreting in a statistical sense the errors in functions of a set of variables, which may be either dependent or independent.

APPENDIX 5

LEAST-SQUARES ADJUSTMENT OF TWO-STATION ANGLE-ONLY POSITION FIXES WITH RELIABILITIES OF ADJUSTED DATA

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LEAST-SQUARES ADJUSTMENT OF TWO-STATION ANGLE-ONLY POSITION FIXES WITH RELIABILITIES OF ADJUSTED DATA

Introduction

One phase of the present program of investigating scoring concepts for weapon-target encounters is concerned with the limitations imposed by errors in the instrumentation. Computational methods making use of redundant data can improve the reliability and usefulness, of the data taken, and this paper is concerned with the least-squares method of adjusting redundant data and with the reliability of adjusted data and of quantities computed from adjusted data.

Certain of the scoring concepts being investigated are based on angle-only observations made from two or more stations separated by base lines of known lengths and orientations. At each such station observations are made of the azimuth and elevation angles of the line of sight to an object whose position is to be determined. It is clear that if simultaneous azimuth and elevation data are available from each of two or more stations, the data are more than the minimum needed to determine the observed object's position, and there is, therefore, redundancy in the data.

The method of least squares applied to redundant angular data gives adjusted observations with minimum variances. It was desired to know more about the extent of the improvement in reliability obtained by adjusting the data and about the least-squares process itself as applied to this particular problem. Since a survey of the literature did not obtain any useful information, a limited investigation was undertaken. This paper is concerned with setting up a least-squares process to adjust the azimuth and elevation angles of an object observed simultaneously from two stations separated by a base line of known length. The angular data are assumed to have equal variances and to be uncorrelated (independent) and unbiased. Expressions are obtained for the separation of the object from either of these observation stations. Expressions for the variances of the adjusted data and of the computed ranges are also developed. It is shown that least-square adjustment of the data reduces the probable errors in both measured and computed quantities by about 50 percent on the average.

One advantage of this process over certain other least-squares procedures is that preliminary estimates of position are not required. The procedures used to adjust the data and to obtain variance estimates for the adjusted observations and computed quantities are straightforward and fairly simple. A disadvantage is that the one condition equation is required to force the adjusted rays from two stations to

intersect, and an additional condition equation would have to be used for each added station; as a result, the adjusted data have non-zero covariances. The process could be readily generalized to include correlated data of differing reliabilities and to include more stations and more observational data; the process is not limited to angular data. For the present purpose of determining the variances of the adjusted data and position fixes, however, these generalizations were not needed and were not developed.

A Condition Equation, Necessary and Sufficient for Rays from Two Stations to Intersect

The elevation and azimuth angular data are assumed to have been corrected for all known errors. These include biases, instrument errors, refraction errors, etc. Since some random error will normally remain uncorrected, the two rays (lines of sight) defined by the corrected angular data will seldom intersect. The data are to be adjusted by the method of least squares to give unbiased mean values of minimum variances, and an additional restricting condition is made that the adjusted rays intersect.

Figure 1 is a sketch of the geometry of the problem. A Cartesian coordinate system is shown; the distance between the observation stations at O and Q on the Y axis is denoted by ℓ. The adjusted position of the observed point is at P where the adjusted rays intersect.

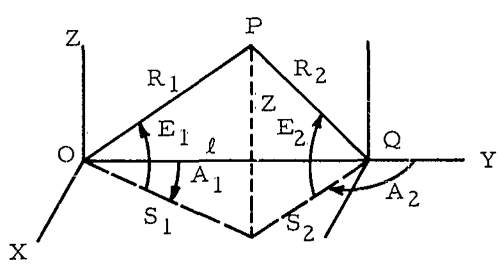


Figure 1

Adjusted Two-Station Geometry

From the geometry,

$$Z = S_1 \tan E_1 = S_2 \tan E_2 \tag{1}$$

By the law of sines,

$$S_1 = \frac{\ell \sin A_2}{\sin (A_2 - A_1)}, \qquad S_2 = \frac{\ell \sin A_1}{\sin (A_2 - A_1)} \tag{2}$$

Substitution of equations (2) into equation (1) gives the condition equation for the adjusted observations,

$$f \equiv \sin A_2 \tan E_1 - \sin A_1 \tan E_2 = 0 \tag{3}$$

Equation (3) may also be obtained by equating the X/Z ratios of the adjusted observed point as measured at the two stations. The foregoing development assumes that the algebraic signs of A_1, A_2, and $(A_2 - A_1)$ are the same; E_1 and E_2 should also have the same signs.

Equation (3) may be shown to be a necessary and sufficient condition for the adjusted rays to intersect, possibly at infinity. The necessity is obvious from the foregoing development. If more than two stations were used, an additional condition equation would be needed for each such additional station. Since these additional restrictions are not needed for the present purposes, they will not be developed.

The Least-Squares Adjustment

Suppose simultaneous azimuth and elevation angles A_1, E_1, A_2, and E_2 are measured at two observation stations as shown in Figure 1. In general, equation (3) is not satisfied exactly, so that a small error remains,

$$f(A_1, E_1, A_2, E_2) = d \tag{4}$$

Using the linear terms of a Taylor's series expansion of f, and representing the changes to be added to the raw data to give the adjusted data as δA_1, δE_1, etc., the linearized condition equation for the adjusted observations is obtained:

$$f = d + \frac{\partial f}{\partial A_1} \delta A_1 + \frac{\partial f}{\partial E_1} \delta E_1 + \frac{\partial f}{\partial A_2} \delta A_2 + \frac{\partial f}{\partial E_2} \delta E_2 = 0 \tag{5}$$

Upon evaluating these partial derivatives at the known approximate values of the angles, equation (5) becomes

$$f = d + a_1\,\delta A_1 + a_2\,\delta E_1 + a_3\,\delta A_2 + a_4\,\delta E_2 = 0 \tag{6}$$

in which

$$\begin{aligned} a_1 &= -\cos A_1 \tan E_2 \\ a_2 &= \sin A_2 \sec^2 E_1 \\ a_3 &= \cos A_2 \tan E_1 \\ a_4 &= -\sin A_1 \sec^2 E_2 \end{aligned} \tag{7}$$

The adjusted observations are related to the observed data as follows:

$$A_{1\ \text{adjusted}} = A_1 + \delta A_1,\ \text{etc.} \tag{8}$$

In most cases of practical interest, the errors in A_1, E_1, A_2, E_2, are uncorrelated and may be considered to have equal variances, σ_0^2. In such cases, the observations all have equal weight, and the least-square problem is to minimize

$$\delta^2 A_1 + \delta^2 E_1 + \delta^2 A_2 + \delta^2 E_2,$$

subject to the linearized condition equation (6).

The easiest way to proceed is to make use of a Lagrangian multiplier, λ, and to minimize

$$H \equiv \delta^2 A_1 + \delta^2 E_1 + \delta^2 A_2 + \delta^2 E_2 - 2\lambda f \tag{9}$$

Upon taking the partial derivatives of H with respect to A_1, E_1, etc., and equating them to zero, the following normal equations result:

$$\begin{aligned} 2\delta A_1 - 2\lambda a_1 &= 0 \\ 2\delta E_1 - 2\lambda a_2 &= 0 \\ 2\delta A_2 - 2\lambda a_3 &= 0 \\ 2\delta E_2 - 2\lambda a_4 &= 0 \end{aligned} \tag{10}$$

The normal equations (10), together with equation (6), may be solved for the unknowns, with the result

$$\begin{aligned} \delta A_1 &= -a_1 d/(a_1^2 + a_2^2 + a_3^2 + a_4^2) \\ \delta E_1 &= -a_2 d/(a_1^2 + a_2^2 + a_3^2 + a_4^2) \\ \delta A_2 &= -a_3 d/(a_1^2 + a_2^2 + a_3^2 + a_4^2) \\ \delta E_2 &= -a_4 d/(a_1^2 + a_2^2 + a_3^2 + a_4^2) \end{aligned} \tag{11}$$

These are the corrections to be applied to the observed data according to equation (8). They should satisfy the linearized condition equation (6), which serves as a check for small corrections. If the corrections should be too large for equation (6), obtained from linear terms in Taylor's series, to be applicable, an iteration process could be used to obtain a better answer.

Reliability of Adjusted Data and Computed Functions

Considerable interest may be attached to the reliabilities of the adjusted data and those of functions computed from these data. In the following it will be presumed that the variance, σ_0^2, of the unadjusted angular measurements is known in advance. However, it will be seen that this value may be estimated from the data.

The adjustments to be made to the data satisfy equation (6). This equation may be written in the form

$$-d = a_1\,\delta A_1 + a_2\,\delta E_1 + a_3\,\delta A_2 + a_4\,\delta E_2 \tag{12}$$

Upon squaring this equation and averaging, there results

$$d^2_{\text{Average}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2)\,\sigma_0^2 \tag{13}$$

This result may be used with equation (11) to obtain the variances of the adjusted observations. Upon squaring equations (11) and averaging, making use of equation (13), one obtains

$$\begin{aligned}
(\sigma^2_{A_1})_{\text{Adjusted}} &= a_1^2\,\sigma_0^2/(a_1^2 + a_2^2 + a_3^2 + a_4^2) \\
(\sigma^2_{E_1})_{\text{Adjusted}} &= a_2^2\,\sigma_0^2/(a_1^2 + a_2^2 + a_3^2 + a_4^2) \\
(\sigma^2_{A_2})_{\text{Adjusted}} &= a_3^2\,\sigma_0^2/(a_1^2 + a_2^2 + a_3^2 + a_4^2) \\
(\sigma^2_{E_2})_{\text{Adjusted}} &= a_4^2\,\sigma_0^2/(a_1^2 + a_2^2 + a_3^2 + a_4^2)
\end{aligned} \tag{14}$$

These results show that on the average the variance of any one of these adjusted observations is only one fourth of the variance of the unadjusted observation, since

$$(\sigma^2_{A_1} + \sigma^2_{E_1} + \sigma^2_{A_2} + \sigma^2_{E_2})_{\text{Adjusted}} = \sigma_0^2$$

while (15)

$$(\sigma^2_{A_1} + \sigma^2_{E_1} + \sigma^2_{A_2} + \sigma^2_{E_2})_{\text{Unadjusted}} = 4\sigma_0^2$$

It may be noted from equation (14) that as any coefficient a_i tends to zero (while at least one of the other $a_j \neq 0$, $j \neq i$), the variance of the corresponding adjusted observed variable tends to zero.

A result of placing the restricting condition, equation (6), on the adjusted variables is that they no longer are independent, but have covariances which are in general different from zero. These covariances may be readily found, using equations (11) and (13). Upon taking the products of pairs of equations listed under (11), averaging, making use of equation (13), there results

$$
\begin{aligned}
\sigma_{A_1E_1} &= a_1a_2\sigma_0^2/(a_1^2 + a_2^2 + a_3^2 + a_4^2) \\
\sigma_{A_1A_2} &= a_1a_3\sigma_0^2/(a_1^2 + a_2^2 + a_3^2 + a_4^2) \\
\sigma_{A_1E_2} &= a_1a_4\sigma_0^2/(a_1^2 + a_2^2 + a_3^2 + a_4^2) \\
\sigma_{E_1A_2} &= a_2a_3\sigma_0^2/(a_1^2 + a_2^2 + a_3^2 + a_4^2) \\
\sigma_{E_1E_2} &= a_2a_4\sigma_0^2/(a_1^2 + a_2^2 + a_3^2 + a_4^2) \\
\sigma_{A_2E_2} &= a_3a_4\sigma_0^2/(a_1^2 + a_2^2 + a_3^2 + a_4^2)
\end{aligned} \tag{16}
$$

This dependence of the adjusted observations is sometimes overlooked in variance computations. The variance of any function of the variables may be computed from their adjusted variances, but the covariance terms should be included in such a computation. The covariance terms are usually smaller than the variance terms and there may be cancellation effects due to algebraic signs. One has always $\sigma_{X_1X_2} = \sigma_{X_1}\sigma_{X_2}\,\rho$, with a correlation coefficient ρ such that $|\rho| \le 1$.

It is interesting to note that the same average value of d^2 is obtained by using either the unadjusted observations or the adjusted observations; this serves as a check on the calculations. It may be noted that the coefficients a_i are simultaneously equal to zero only if the observed point lies along the Y axis. In this case no change in the observations is made by the least-squares adjustment. Also, the two rays intersect at an indeterminate point along the Y axis, and, as will be seen from later equations, the variance of the range is infinite. This indicates that one should strive to have the observed point not located along the Y axis or even close to it.

Expressions for the range may be obtained by the use of trigonometry. By use of the law of sines for triangle POQ in Figure 1, one obtains

$$R_1 = \frac{\ell\sqrt{1 - \cos^2 E_2 \cos^2 A_2}}{\cos A_2 \cos E_2 \sqrt{1 - \cos^2 A_1 \cos^2 E_1} + \cos A_1 \cos E_1 \sqrt{1 - \cos^2 A_2 \cos^2 E_2}} \tag{17}$$

with a similar expression for R_2. Alternate forms are less complicated:

$$\begin{aligned} R_1 &= \ell \sin A_2 / \cos E_1 \sin (A_2 - A_1) \\ R_2 &= \ell \sin A_1 / \cos E_2 \sin (A_2 - A_1) \end{aligned} \tag{18}$$

The differentials of equations (18) may be taken, with the following results:

$$\begin{aligned} dR_1 &= M_1 dA_1 + M_2 dE_1 + M_3 dA_2 \\ dR_2 &= N_1 dA_1 + N_2 dE_2 + N_3 dA_2 \end{aligned} \tag{19}$$

where

$$\begin{aligned} M_1 &= \ell \sin A_2 \csc (A_2 - A_1 \cot (A_2 - A_1) \sec E_1 \\ M_2 &= \ell \sin A_2 \csc (A_2 - A_1) \sec E_1 \tan E_1 \\ M_3 &= \ell \sec E_1 \csc (A_2 - A_1) [\cos A_2 - \sin A_2 \cot (A_2 - A_1)] \\ N_1 &= \ell \sec E_2 \csc (A_2 - A_1) [\cos A_1 + \sin A_1 \cot (A_2 - A_1)] \\ N_2 &= \ell \sin A_1 \csc (A_2 - A_1) \sec E_2 \tan E_2 \\ N_3 &= -\ell \sin A_1 \sec E_2 \csc (A_2 - A_1) \cot (A_2 - A_1) \end{aligned} \tag{20}$$

Upon squaring and averaging equations 19, the following variances result:

$$\begin{aligned} (\sigma^2_{R_1})_{\text{Adjusted}} = \Big[& M_1^2 \sigma^2_{A_1} + M_2^2 \sigma^2_{E_1} + M_3^2 \sigma^2_{A_2} + 2M_1 M_2 \sigma_{A_1 E_1} \\ & + 2M_1 M_3 \sigma_{A_1 A_2} + 2M_2 M_3 \sigma_{E_1 A_2} \Big]_{\text{Adjusted}} \end{aligned} \tag{21}$$

$$(\sigma^2_{R_2})_{\text{Adjusted}} = \left[N_1^2 \sigma^2_{A_1} + N_2^2 \sigma^2_{E_2} + N_3^2 \sigma^2_{A_2} + 2N_1 N_2 \sigma_{A_1 E_2} \right.$$

$$\left. + 2N_1 N_3 \sigma_{A_1 A_2} + 2N_2 N_3 \sigma_{E_2 A_2} \right]_{\text{Adjusted}}$$

Since the range is computed from the adjusted variables, its variance will be approximately proportional to the variances of the adjusted variables. The variance of the range is thus on the average approximately one fourth of what it would be if the range were computed from the unadjusted variables; the standard error in range is reduced by a factor of about one half. Cartesian coordinates of the observed object could of course be computed from the spherical-polar coordinates, and their variances could be readily obtained by the use of differentials.

APPENDIX 6

APPLICATION OF POLYNOMIAL-BASED SMOOTHING AND INTERPOLATING FORMULAS TO SCORING PROBLEMS

APPENDIX 6

APPLICATION OF POLYNOMIAL-BASED SMOOTHING AND INTERPOLATING FORMULAS TO SCORING PROBLEMS

Many formulas are available for smoothing and interpolating data taken at a constant time interval. Most of these formulas are based on the premise that the function of the time represented by the data on which these operations are carried out is given with sufficient accuracy over a short interval by a polynomial of a specified low degree. Since the results will not be satisfactory in general unless this assumption is valid, it is desirable to determine the true nature of the function represented by the data. The purpose of this paper is to show that polynomial-based formulas are suitable for use on certain types of data obtained for scoring the attack of a missile on a target.

With trajectories of long duration, it may be difficult to specify a priori the probable character of a variable function of the time. However, with trajectories of short duration, where accelerations have limited time to affect the variables, the probable behavior of a variable may often be inferred. Thus a linear relative trajectory provides a good approximation for a relative trajectory of sufficiently short duration. By a linear relative trajectory is meant a trajectory of the weapon relative to the target which is given by a linear vector function of the time,

$$\vec{R} = \vec{R}_o + \dot{\vec{R}}_o t \qquad (1)$$

with $\vec{R}_o$ and $\dot{\vec{R}}_o$ constant vectors. The addition of a small non-zero acceleration term to the right side of equation (1) would not affect the conclusion to be drawn in the following discussion.

The components of equation (1) in a rectangular Cartesian coordinate system are as follows:

$$\begin{aligned} X &= X_o + \dot{X}_o t \\ Y &= Y_o + \dot{Y}_o t \\ Z &= Z_o + \dot{Z}_o t \end{aligned} \qquad (2)$$

For short-duration trajectories there is considerable probability that the relative Cartesian coordinates of the weapon with respect to the target are well approximated by low-degree polynomials in the time. Polynomial-based smoothing and interpolating formulas should thus be suitable for application to the relative Cartesian coordinates X, Y, Z.

The case is different for relative polar coordinates, as will be seen. Let the polar coordinates, range, azimuth, and elevation, be considered as functions of the time, considering first the range, R.

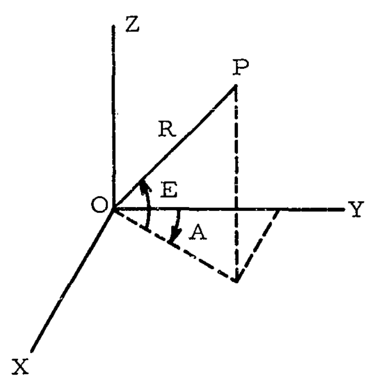
$$R^2 = \vec{R} \cdot \vec{R} \cong R_o^2 + 2\vec{R}_o \cdot \dot{\vec{R}}_o t + \dot{R}_o^2 t^2 \tag{3}$$

$$|R| = (R^2)^{\frac{1}{2}} \cong (a + bt + ct^2)^{\frac{1}{2}} \tag{4}$$

Equations (3) and (4) show that R^2 for a linear relative trajectory may be well approximated by a polynomial of low degree while R may not, in general. This suggests that better results should be obtained if R^2 rather than R is smoothed or interpolated. If R is smoothed or interpolated, good results should be obtained only for $\dot{\vec{R}}_o$ and $\vec{R}_o$ closely aligned.

The figure shows the non-rotating Cartesian coordinate system X, Y, Z with origin O at the target. The weapon is at P, and it will be seen that the azimuth and elevation angles are given by

$$\begin{aligned} \tan A &= \frac{X}{Y} \sim \frac{d + et}{f + gt} \\ \sin E &= \frac{Z}{R} \sim \frac{h + kt}{(a + bt + ct^2)^{\frac{1}{2}}} \end{aligned} \tag{5}$$



Thus A and E are not given even approximately by low-degree polynomials in t over any appreciable time interval. If the variables A and E are observed, and are operated on with polynomial-based smoothing or interpolation formulas, the results should not be expected to be very good over appreciable time intervals.

The angular rates $\dot{A}$ and $\dot{E}$ of $\vec{R}$ may be determined by differentiation of equations (5). It is found that $\dot{A} \sim \frac{K}{M + Nt + Qt^2}$. Thus, $1/\dot{A}$ should be given approximately by a low order polynomial in t, and for best results $1/\dot{A}$ rather than $\dot{A}$ should be used with polynomial-based smoothing or interpolating formulas. On the other hand, neither $\dot{E}$ nor $1/\dot{E}$ appears to be given approximately by a low-degree polynomial in t over any appreciable time interval. However, for $\vec{R}_o$ an·l $\dot{\vec{R}}_o$ somewhat closely aligned, the quantity (sec E)/$\dot{E}$ would be give . approximately by a low-degree polynomial in t, so that operating on this quantity should be expected to yield more valid results than operating on E or $\dot{E}$ alone. It may also be shown that ($\sec^2$ E)/$\dot{A}$ should be closely approximated by a low-degree polynomial in t for linear relative trajectories, even with $\vec{R}_o$ and $\dot{\vec{R}}_o$ not closely aligned.

In conclusion, it appears probable that relative-Cartesian-coordinate data for a short-duration weapon-target encounter may be well adapted to numerical methods of extrapolating, interpolating, and smoothing, whereas relative-polar-coordinate data does not appear to be so well adapted. A reasonable procedure for performing these operations on the relative polar coordinates would be to first convert the data to Cartesian coordinates, perform the operations required, and then convert the resultant values to polar coordinates. Certain functions of the relative polar coordinates have been shown to be well approximated by polynomials (in the time) of low degree and are therefore well adapted to the numerical operations of smoothing, interpolation, and extrapolation.

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